GEOMETRY TMS EXAM
October 01, 2015

Duration: 3 hours.

(1) Let $f : \mathbb{R}^3 \mapsto \mathbb{R}^4$ be the map defined by $f(x, y, z) = (x^2 - y^2, xy, xz, yz)$. Consider $\mathbb{R}P^2$ as $S^2/\sim$ where $p \sim -p$ for all $p \in S^2$.
a) Write down a chart for $\mathbb{R}P^2$.
b) Let $F : \mathbb{R}P^2 \mapsto \mathbb{R}^4$ induced by f. Find F_*.
c) Is F embedding? Why?

(2) a) Show that the set $SL(2, \mathbb{R})$ of 2×2 real matrices whose determinant is equal to 1 is a submanifold of $\mathbb{R}^4$. What is its dimension?
b) Prove that the tangent space to $SL(2, \mathbb{R})$ at the identity matrix $A = I$ may be identified with the set of matrices of zero trace.

(3) Let M be an even dimensional manifold, $dim M = 2n$. A differential form $\omega \in \Omega^2(M)$ is said to be non-degenerate if

$$\wedge^n \omega := \omega \wedge \cdots \wedge \omega \in \Omega^{2n}(M)$$

is a volume form. Show that on a compact orientable manifold M without boundary a non-degenerate 2-form ω cannot be exact.

(4) Let $\omega = \dfrac{xdy - ydx}{2\pi} \in \Omega^1(\mathbb{R}^2)$ and $f : S^1 \longrightarrow S^1$ defined by $f(z) = z^k$, $k \in \mathbb{Z}_+$. Calculate

$$\int_{S^1} f^*(w).$$

(5) On $\mathbb{R}^4$ with coordinates (x, y, z, w) consider the following vector fields; $X_1 = x\dfrac{\partial}{\partial y} - y\dfrac{\partial}{\partial x}$ and $X_2 = y\dfrac{\partial}{\partial z} - z\dfrac{\partial}{\partial y}$ and 2-form $\omega = xdx \wedge dy + zdz \wedge dw$. Compute the following:
a) $[X_1, X_2]$
b) $d\omega$
c) $\Phi^*(\omega)$ where $\Phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^4$ is the map $\Phi(t, u) = (t \cos t, u, t \sin t, u)$.