M.E.T.U.
Department of Mathematics
TMS Exam in Geometry
September 18$^{\text{th}}$, 2019
Duration: 180 minutes

There are 5 problems and each is worth 20 points.

1. Let $M = \{(x, y, z) \in \mathbb{R}^3 : (1 - z^2)(x^2 + y^2) = 1\}$
(a) Show that M is an embedded smooth submanifold of $\mathbb{R}^3$.
(b) Show that the restriction of vector field $V = z^2x\frac{\partial}{\partial x} + z^2y\frac{\partial}{\partial y} + z(1-z^2)\frac{\partial}{\partial z}$ on $\mathbb{R}^3$ is a tangent vector field of M.

2. Let S^1 be the unit circle in $\mathbb{R}^2$ and $t = \dfrac{x}{1-y}$ be the coordinate on S^1 given by stereographic projection from north pole, N. Let $V = v(t)\frac{d}{dt}$ denote a smooth vector field on $S^1\backslash\{N\}$.
(a) Express V on $S^1\backslash\{N, S\}$ by using the coordinate on S^1 given by stereographic projection from south pole, S.
(b) Find a condition on $v(t)$ that guarantees that V extends to a global smooth vector field on S^1. Does $V = t^3\frac{d}{dt}$ extend to a smooth vector field on S^1?

3. Consider the following vector fields on $\mathbb{R}^3$

$$V = x\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + 2\frac{\partial}{\partial z}$$
$$W = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

Compute the following quantities.

(a) The curve $\beta(t)$ in $\mathbb{R}^3$ such that $\beta'(t) = V(\beta(t))$, $\beta(0) = (x_0, y_0, z_0)$ i.e. the integral curve of V.
(b) The push-forward $F_*W_{|(u,v,w)}$ where $F : \mathbb{R}^3\backslash\{y = 0\} \longrightarrow \mathbb{R}^3$ defined as

$$F(x, y, z) = (y - x, 1 - \frac{x}{y}, 2z)$$

(c) The Lie bracket $[V, W]$.

4. (a) Prove that a submersion $F : M \longrightarrow N$ is an open map.
(b) Show that there is no submersion $F : S^3 \longrightarrow \mathbb{R}^2$. How about if $\mathbb{R}^2$ is replaced by S^2?

5. Let $i : S^1 \longrightarrow \mathbb{R}^2$ be the inclusion of unit circle S^1 into $\mathbb{R}^2$ with standard orientation and ω be the closed 1-form $\omega = \dfrac{xdy - ydx}{x^2 + y^2}$ on $\mathbb{R}^2$.

(a) Compute $\displaystyle\int_{S^1} i^*\omega$.

(b) Is $i^*\omega$ exact on S^1?

(c) Show that on a compact manifold M a non-zero (non-zero at every point of M) 1-form can not be exact.

(d) Let M be a smooth compact manifold which admits a smooth submersion $F : M \longrightarrow S^1$. Use previous parts to show that there exists a closed 1-form on M which is not exact (i.e. $H^1_{DR}(M) \neq 0$).