Differentiable Manifolds
TMS EXAM
11 February 2013

Duration: 3 hr.

1. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = x^3 + xy + y^3 + 1 \ .$$

For which of the points $p = (0, 0)$, $p = (1/3, 1/3)$, $p = (-1/3, -1/3)$ is $f^{-1}(f(p))$ an imbedded submanifold in $\mathbb{R}^2$?

2. Let M be the hyperboloid of two sheets given by $y^2 - z^2 - x^2 = 1$.

(**a**) Let $p \in M$. Explain how we can identify T_pM by a subspace of $\mathbb{R}^3$ using a chart at p.

(**b**) Describe $T_p(M)$ as a subspace of $\mathbb{R}^3$ if $p = (0, 2, \sqrt{3})$.

(**c**) Determine whether the map which assigns to each point $q = (x, y, z)$ the vector $(y, x+z, y)$ is a smooth vector field on M.

3. Let $F : M \to N$ be a smooth function between the manifolds M and N and let a be a smooth function on M.

(**a**) Show that $F^*(da) = d(F^*(a))$

(**b**) Verify the formula $F^*d = dF^*$ on the forms of type $\phi_1 \wedge \phi_2$ where ϕ_1 and ϕ_2 are 1-forms.

(**c**) Let $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$g(x, y, z) = (xy, x^2yz)$$

Compute $g^*(2xydx \wedge dy)$

4. Let

$$\alpha = \frac{1}{2\pi}\frac{xdy - ydx}{x^2 + y^2}$$

(**a**) Prove that α is a closed 1-form on $\mathbb{R}^2 \setminus 0$

(**b**) Compute the integral of α over the unit circle S^1 ?

(**c**) How does this shows that α is not exact?