Differentiable Manifolds
TMS EXAM
13 February 2015

Duration: 3 hr.

1. Let S^2 be the unit sphere in $\mathbb{R}^3$. Consider it with the topology relative to $\mathbb{R}^3$. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion map.

(**a**) Show that i is an immersion.

(**b**) Is i an embedding? Why?

2. Let M, N be two differentiable manifolds and $f : M \to N$ be a smooth map. Define a new map $F : M \to M \times N$ by $F(p) = (p, f(p))$.

(**a**) Show that F is smooth.

(**b**) Show that $F_*(v) = (v, f_*(v))$ where F_* and f_* are induced maps at a point p of M and v is a tangent vector of M at p.

(**c**) Show that the tangent space to graph(f) at the point $(p, f(p))$ is the graph of $f_* : T_pM \to T_{f(p)}N$

3. Consider the 1-form $w = (x^2 + 7y)dx + (-x + y\sin \mathrm{y}^2)\mathrm{dy}$ on $\mathbb{R}^2$.

(**a**) Is w exact? Is it closed?

(**b**) Compute the integral of w over each side of the triangle whose vertices are $(0, 0), ((1, 0), (0, 2)$ where the sides are oriented in such a way that the triangle is oriented counterclockwise.

4. Let F: $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be the map $F(p) = -p$.

(**a**) What is the induced map F_*? Why?

(**b**) Show that antipodal map $A : S^n \to S^n$ which is the restriction of F on the n-sphere is orientation preserving if and only if n is odd.

(**c**) Prove that the real projective space $\mathbb{R}P^n$ is orientable if and only if n is odd.