METU Department of Mathematics

TMS EXAM IN ~~TOPOLOGY~~ Geometry
(February , 2018, 10:00–13:00)

1. (a) Tangent vector $v \in T_{(1,1)}\mathbb{R}^2$ at point $(1,1) \in \mathbb{R}^2$ in Cartesian coordinates (x,y) can be expressed as $2\frac{\partial}{\partial x} + 1\frac{\partial}{\partial y}$. Express vector v in the polar coordinates (r,θ) in terms of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial \theta}$.

(b) Let α denote the restrictions to $S^2 = \{x^2+y^2+z^2 = 1\} \subset \mathbb{R}^3$ of the differential forms dx in $\mathbb{R}^3$. At which points p of S^2 the differential forms α vanishes as a linear map $\alpha_p \colon T_pS^2 \to \mathbb{R}$?

(c) Express 2-form $dz \wedge dx$ in chart (x,y) on the bottom half-sphere $S^2_- = \{x^2+y^2+z^2 = 1, z < 0\}$.

2. Determine if the following maps are immersions, submersions, embedding, or neither.

(a) $\mathbb{R} \to \mathbb{R}^2$, $t \mapsto (\cos t, \sin(2t+1))$.

(b) $\mathbb{R}^2 \to \mathbb{R}$, $(x,y) \mapsto x\cos y$.

(c) $(0,1)\times(0,1) \to \mathbb{R}^2$, $(x,y) \mapsto (\cos x, \sin y)$.

3. For the function $f(x,y) = x^2y$ and vector fields $V = x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}$, and $U = (x+y+1)\frac{\partial}{\partial x} + (xy)\frac{\partial}{\partial y}$ in $\mathbb{R}^2$ find:

(a) $V(f)$ (differentiate f with respect to vector field V),

(b) $df(U)$ (the differential of f on the vector field U),

(c) $[V,U]$ (the Lie bracket of V and U),

(d) L_Vf, $L_V(df)$, $L_V(dx\wedge dy)$ and $L_V(W)$ (the Lie derivatives of f, df, $dx \wedge dy$ and W with respect to vector field V).

4. Assume that $f(x,y) = 0$ defines in $\mathbb{R}^2$ a smooth curve C.

(a) Prove that the restrictions to C of the 1-forms $\frac{dx}{f_y}$ and $-\frac{dy}{f_x}$ coincide at every point where the partials f_x and f_y do not vanish.

(b) Deduce that the above 1-form can be extended to the whole curve C.

(c) Conclude that the above 1-form does not vanish at all points on curve C.