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Department of Mathematics
TMS Exam in Geometry
March 12$^{\text{th}}$, 2021
Duration: 180 minutes

1. Consider the set S of all straight lines in $\mathbb{R}^2$ (not necessarily through the origin). Show that S is a smooth manifold by giving a smooth structure.

(*Note that for* $(a_1, b_1) \neq (0,0)$ *and* $(a_2, b_2) \neq (0,0)$, $a_1x+b_1y+c_1 = 0$ *and* $a_2x+b_2y+c_2 = 0$ *define the same line if and only if* $(a_1, b_1, c_1) = \lambda(a_2, b_2, c_2)$ *for some non-zero* $\lambda \in \mathbb{R}$.)

2. Consider the sets $M_a = \{(x, y, z) \in \mathbb{R}^3 \, : \, x^2 + y^2 + z^2 = a\}$ and $N_b = \{(x, y, z) \in \mathbb{R}^3 \, : \, x - y^2 = b\}$.

(a) Show that $M_2 \cap N_0 = \{(x, y, z) \in \mathbb{R}^3 \, : \, x^2 + y^2 + z^2 = 2, \;\; x - y^2 = 0\}$ is a submanifold of $\mathbb{R}^3$. What is the dimension of this submanifold?

(b) Write a basis for the tangent space $T_{(1,1,0)}(M_2 \cap N_0) \subset T_{(1,1,0)}\mathbb{R}^3$.

(c) For which values of a and b is $M_a \cap N_b$ a submanifold of $\mathbb{R}^3$? Explain your answer.

3. Consider $F : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ defined as

$$F(x, y, z) = (x, y + x^2, z - xy) = (u, v, w)$$

Consider the vector fields

$$X = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$$
$$Y = -x^2\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$$

and 1-form $\varphi = vu^2du + wdv + e^u dw$

(a) Compute $[F_*X, F_*Y]$ where F_* is the push-forward/induced map which is defined as $(F_*V)(p) = F_*(V_p)$ for any vector field V.

(b) Verify that $d(F^*\varphi) = F^*(d\varphi)$.

4. Let S be the portion of the cylinder $S: \ x^2 + y^2 = 1$ consisting of points with $0 \leq z \leq 1, \ x > 0$.
(a) Choose the orientation on S assigning to each point $p = (a, b, c)$ on M the orientation class $[(-b, a, 0), (0, 0, 1)]$ of the tangent space $T_p(S)$. Exhibit a parametrization $\phi : U \to S$, $U \subset \mathbb{R}^2$, about p so that ϕ is orientation preserving.

(b) What is the boundary ∂S of S? Let $p = (a, b, c)$ be a point on ∂S, exhibit a basis for $T_p(\partial S)$.

(c) Describe the boundary orientation on ∂M explicitly when S is given the orientation on part (a).