September, 2016

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. (a) Suppose $A \in \mathbb{C}^{m\times m}$ and $\lambda_1, \cdots, \lambda_m \in \mathbb{C}$ are eigenvalues of A. Prove that

$$tr(A) = \sum_{j=1}^{m} \lambda_j$$

   (b) Prove that, if A is Hermitian, then there is a unitary matrix U and a diagonal matrix D such that $U^*AU = D$

   (c) Prove that, if A is a real symmetric matrix, then there is an orthogonal matrix O and a diagonal matrix D such that $O^TAO = D$

2. We wish to solve $Ax = b$ iteratively where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Show that for this A the Jacobi method and the Gauss-Seidel method both converge. Explain why for this A one of these methods is better than the other.

3. Suppose $A \in \mathbb{R}^{n\times n}$ and $\|\cdot\|$ denotes a matrix norm (Not necessarily induced by a vector norm) which also satisfies the compatibility property: for all $B \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{n\times n}$,

$$\|BC\| \leq \|B\|\|C\|.$$

Let $\rho(A)$ denote the spectral radius of A.

   (a) Show that for $\rho(A) \leq \|A\|$ and then, that $\rho(A) \leq \|A^k\|^{1/k}$.

   (b) Show that for any given $0 < \epsilon << 1$,

$$\lim_{k\to\infty} \frac{\|A^k\|}{(\rho(A)+\epsilon)^k} = 0.$$

   Thus, conclude that $\lim_{k\to\infty} \|A^k\|^{1/k} \leq \rho(A)$.
   (Hint: You might use the fact that there exists an operator norm $\|A\|_{\epsilon,A}$ such that $\|A\|_{\epsilon,A} \leq \rho(A) + \epsilon/2$, for all $\epsilon > 0$)

   (c) From parts (a) and (b) what can you say about

$$\lim_{k\to\infty} \|A^k\|^{1/k}.$$

4. Let A be a real symmetric $n \times n$ matrix having the eigenvalues λ_1 with

$$|\lambda_1| > |\lambda_2| \geq \cdots \geq |\lambda_n|$$

and the corresponding eigenvectors $x_1, \cdots, x_n$ with $\chi_i^T \chi_k = \delta_{ik}$. Starting with an initial vector y_0 for which $x_i^T y_0 \neq 0$, suppose one computes

$$y_{k+1} := \frac{1}{\| Ay_k \|} Ay_k \quad \text{for} \quad k = 0, 1, 2, \cdots$$

with an arbitrary vector norm $\| \cdot \|$, and concurrently the quantities

$$q_{ki} := \frac{(Ay_k)_i}{(y_k)_i}, \quad 1 \leq i \leq n, \quad \text{in case } \ (y_k)_i \neq 0,$$

and the Rayleigh quotient

$$r_k := \frac{y_k^T A y_k}{y_k^T y_k}$$

Prove the following:

(a) $q_{ki} = \lambda_1[1 + O((\lambda_2/\lambda_1)^k)]$ for all i with $(x_1)_i \neq 0$.

(b) $r_k = \lambda_1[1 + O((\lambda_2/\lambda_1)^{2k})]$.