September, 2024

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the following matrix and vector:

$$A = \begin{bmatrix} 2 & 1 \\ 2 & -1 \\ 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) Descibe the singular value decomposition (SVD) of an $m \times n$ matrix.

   (b) Find the SVD of the matrix A.

   (c) Compute the pseudo-inverse of A using the SVD obtained in Part (b).

   (d) Find the least squares solution of the minimum norm for the system $Ax = b$.

2. Given a square matrix A, one can construct an RQ-factorization of A with R is an upper triangular and Q is orthogonal. This is essentially the same as the QR factorization but the elementary transformations are applied from the right instead of the left.

   Consider the explicitly shifted RQ algorithm:

   Assume $AV = VH$ (orthogonal reduction to upper Hessenberg form). Then, the main iterative loop of the explicitly shifted RQ-iteration will be:

   - for $j = 1, 2, \cdots$

     $\mu \leftarrow \mu_j$ (choose a shift)
     Factorize $H - \mu I = RQ$
     $H \leftarrow QR + \mu I$
     $V \leftarrow VQ^T$

   - end

   (a) Show $H_+ = QR + \mu I$ is similar to H (thus similar to A) throughout the iteration, i.e. show $H_+ Q = QH$.

   (b) Show H remains upper Hessenberg throughout the iteration.

   (c) Prove that $(A - \mu I)v_1^+ = v_1 \rho_{11}$, where $v_1^+ = V_+ e_1$, with $V_+ = VQ^T$ and where ρ_{11} is the first diagonal element of R.

   (d) What choice of μ_j would give a Rayleigh quotient shift for this iteration?

3. Let $Ax = b$, where $A \in \mathcal{R}^{n\times n}$ is non-zero and $x \in \mathcal{R}^n$ is non-zero.

   (a) Show that for any vector $y \in \mathcal{R}^n$ and any p-norm

   $$\frac{\| r \|_p}{\| A \|_p \| x \|_p} \leq \frac{\| x - y \|_p}{\| x \|_p} \leq \kappa_p(A) \frac{\| r \|_p}{\| A \|_p \| x \|_p},$$

   where $r = b - Ay$ denotes the residual and $\kappa_p(A)$ is the condition number of A in the p-norm. Interpret this result.

   (b) Suppose

   $$(A + \Delta A)y = b\Delta b$$

   where $\Delta A \in \mathcal{R}^{n\times n}$, $\Delta b \in \mathcal{R}^n$, with $||\Delta A|| \leq \epsilon ||A||$ and $||\Delta b|| \leq \epsilon ||b||$.
   If $\epsilon\kappa(A) = s < 1$, one can show(you don't need to prove) that $A + \Delta A$ is nonsingular, and

   $$\frac{||y||}{||x||} \leq \frac{1+s}{1-s}.$$

   Use that result to prove

   $$\frac{||y - x||}{||x||} \leq \frac{2\epsilon}{1-s}\kappa(A).$$

   What does this result mean for an ill-conditioned system?

4. Consider the matrix $A = \begin{pmatrix} 9 & -3 & -3 \\ -3 & 10 & 1 \\ -3 & 1 & 5 \end{pmatrix}$

   (a) Show that A has a unique Cholesky factorization, without computing it.

   (b) Compute the Cholesky factorization of A, and use it to solve the linear system $Ax = b$ with $b = (-9, -1.5, 5)^T$.

   (c) Perform one iteration of the SOR method with relaxation parameter $\omega = 1.1$ for the linear system $Ax = b$ from (b). Use the starting point $x^{(0)} = (0, 0, 0)^T$. Does the iterations converge towards the solution? Explain.