February, 2014

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Prove that for invertible square real matrices A and B, the followings hold

   (a) $\|A^{-1} - B^{-1}\| \leq \|A^{-1}\| \|B - A\| \|B^{-1}\|$.

   (b) If $B = A + \delta A$ is the perturbed matrix and $\|\delta A\| \|B^{-1}\| = \delta < 1$ then

   $$\|A^{-1}\| \leq \frac{1}{1-\delta} \|B^{-1}\|$$

   and

   $$\|A^{-1} - B^{-1}\| \leq \frac{\delta}{1-\delta} \|B^{-1}\| .$$

   (c) If $x = A^{-1}b$ and $x + \delta x = (A + \delta A)^{-1}b$ then

   $$\|\delta x\| \leq \frac{\delta}{1-\delta} \|x + \delta x\|$$

   $$\|\delta x\| \leq \frac{\epsilon}{1-\epsilon} \|x\|$$

   where

   $$\delta = \|\delta A\| \|B^{-1}\| < 1, \qquad \epsilon = \|\delta A\| \|A^{-1}\| < 1.$$

   (Note: The norms for matrices and vectors are any compatible matrix and vector norms.)

2. (a) In iteratively solving the linear system $Ax = b$ ($\det A \neq 0$), we generate a sequence $x^{(k)}$ by the formula

   $$x^{(k+1)} = x^{(k)} + wP^{-1}r^{(k)}$$

   starting with some initial guess $x^{(0)}$. Here, P is a nonsingular matrix, $w > 0$ be a constant and $r^{(k)} = b - Ax^{(k)}$ is the residual vector. Show that the method converges if $w|\lambda|^2 < 2\alpha$ for any complex eigenvalue $\lambda = \alpha + i\beta$ of $P^{-1}A$.

   (b) Obtain a convergent iterative sequence using the method given in part (a) with a suitable choice of w for solving $Ax = b$ (carry out 2 iterations with $x^{(0)} = [0,\ 0]^T$ and P is the identity matrix) where

   $$A = \begin{bmatrix} 0 & 2 \\ -1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

3. Let $\{p_j(x)\}_{j=0}^{k}$ be the set of orthonormal polynomials (j-th degree) on an interval $[a,b]$ with the inner product $< h, g >= \int_a^b h(x)g(x)dx$ for continuous functions h and g on $[a,b]$ and the norm is

$$\|g\| =< g, g >^{1/2} .$$

If $p_k^*(x) = \sum_{j=0}^{k} < f, p_j > p_j(x)$ is the best least squares approximation to a continuous function $f(x)$ on $[a,b]$, then show that

(a) $\lim_{k\to\infty} \|f - p_k^*\| = 0,$

(b) $\|f - p_n^*\|^2 = \displaystyle\sum_{j=n+1}^{\infty} < f, p_j >^2.$

4. Let

$$\begin{pmatrix} 8 & 1 & 0 \\ 1 & 4 & \epsilon \\ 0 & \epsilon & 1 \end{pmatrix}, \qquad -1 < \epsilon < 1.$$

(a) Find LU factorization of A.

(b) Find Cholesky factorization of A, if any exits.

(c) Give estimates based on Gerschgorin's theorem for the eigenvalues of A.

(d) Show that it is positive definite.