February, 2024

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Consider the linear system $Ax = b$ where

$$A = \begin{bmatrix} 9 & \alpha & -3 \\ \alpha & 10 & 1 \\ -3 & 1 & 2+\beta \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}$$

with real numbers α and β.

(a) Let $\alpha = -3$. Find the value(s) of β, if any, such that the matrix A has a Cholesky factorization without computing it.

(b) Compute the Cholesky factorization of the matrix A when $\alpha = -3$ and $\beta = 3$, and use it to solve the linear system $Ax = b$.

(c) Find the conditions on the variables α and β for which the Jacobi method and the Gauss-Seidel method converge to the true solution of the system $Ax = b$?

(d) Perform one iteration of the Gauss-Seidel method with $\alpha = -3$ and $\beta = 3$ for the linear system $Ax = b$, by using the initial approximation $x_0 = [0, 1, 0]^T$.

2. Let A be $m \times n, (m \geq n)$ real matrix and let A have rank n. Let $b \in \mathbb{R}^m$. Prove the following statements.

(a) There exists unique $x \in \mathbb{R}^n$ minimizing $\|Ax - b\|_2^2$ where $\|\cdot\|_2$ is the 2-norm.

(b) The matrix $A^T A$ is invertible and $x = (A^T A)^{-1} A^T b$.

3. Given $A \in \mathbb{R}^{n\times n}$ such that

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \vdots & & \ddots & \ddots & & & & \vdots \\ \vdots & & & \ddots & \ddots & & & \vdots \\ \vdots & & & & \ddots & \ddots & & \vdots \\ \vdots & & & & & \ddots & 1 & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & \cdots & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}.$$

(a) Find the Gershgorin's disks for the eigenvalues λ of A.

(b) If $n = 5$ and $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, $a_3 = 0$, $a_4 = -2$, **draw** the Gerschgorin's disks and **find** a region that contains all the eigenvalues of A.

4. Let $\mathbf{UDV^T}$ be the Singular Value Decomposition (SVD) of an $\mathbf{m} \times \mathbf{n}$ matrix $\mathbf{A}$. Prove that

$$\left\|\mathbf{U^T A V}\right\|_{\mathbf{F}}^{\mathbf{2}} = \sum_{\mathbf{i=1}}^{\mathbf{p}} \sigma_{\mathbf{i}}^{\mathbf{2}}$$

where $\mathbf{p} = \min\{\mathbf{m}, \mathbf{n}\}$, $\sigma_{\mathbf{i}}$ denote the singular values of $\mathbf{A}$ and $\|\cdot\|_{\mathbf{F}}$ denotes the Frobenius norm (or Euclidean norm) defined by

$$\|\mathbf{A}\|_{\mathbf{F}} = \left[\sum_{\mathbf{j=1}}^{\mathbf{n}} \sum_{\mathbf{i=1}}^{\mathbf{m}} |\mathbf{a_{ij}}|^{\mathbf{2}}\right]^{\mathbf{1/2}}$$

with $\mathbf{a_{ij}}$ are the entries of the matrix $\mathbf{A}$.