1. Consider the linear system $Ax = b$ where

$$
A = \begin{bmatrix}
9 & \alpha & -3 \\
\alpha & 10 & 1 \\
-3 & 1 & 2 + \beta
\end{bmatrix}, \quad b = \begin{bmatrix}
6 \\
1 \\
-2
\end{bmatrix}
$$

with real numbers $\alpha$ and $\beta$.

(a) Let $\alpha = -3$. Find the value(s) of $\beta$, if any, such that the matrix $A$ has a Cholesky factorization without computing it.

(b) Compute the Cholesky factorization of the matrix $A$ when $\alpha = -3$ and $\beta = 3$, and use it to solve the linear system $Ax = b$.

(c) Find the conditions on the variables $\alpha$ and $\beta$ for which the Jacobi method and the Gauss-Seidel method converge to the true solution of the system $Ax = b$?

(d) Perform one iteration of the Gauss-Seidel method with $\alpha = -3$ and $\beta = 3$ for the linear system $Ax = b$, by using the initial approximation $x_0 = [0, 1, 0]^T$.

2. Let $A$ be an $m \times n$ ($m \geq n$) real matrix and let $A$ have rank $n$. Let $b \in \mathbb{R}^m$. Prove the following statements.

(a) There exists unique $x \in \mathbb{R}^n$ minimizing $\|Ax - b\|_2^2$ where $\|\cdot\|_2$ is the 2-norm.

(b) The matrix $A^TA$ is invertible and $x = (A^TA)^{-1}A^Tb$.

3. Given $A \in \mathbb{R}^{n \times n}$ such that

$$
A = \begin{bmatrix}
0 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\
0 & 0 & 1 & 0 & \cdots & \cdots & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & \cdots & 0 & 1 \\
-a_0 & -a_1 & -a_2 & \cdots & \cdots & \cdots & -a_{n-2} & -a_{n-1}
\end{bmatrix}
$$

(a) Find the Gershgorin’s disks for the eigenvalues $\lambda$ of $A$.

(b) If $n = 5$ and $a_0 = 1$, $a_1 = 0$, $a_2 = -1$, $a_3 = 0$, $a_4 = -2$, draw the Gershgorin’s disks and find a region that contains all the eigenvalues of $A$.

4. Let $UDV^T$ be the Singular Value Decomposition (SVD) of an $m \times n$ matrix $A$. Prove that

$$
\left\| U^TAV \right\|_F^2 = \sum_{i=1}^{\min\{m,n\}} \sigma_i^2
$$

where $p = \min\{m,n\}$, $\sigma_i$ denote the singular values of $A$ and $\|\cdot\|_F$ denotes the Frobenius norm (or Euclidean norm) defined by

$$
\|A\|_F = \left[ \sum_{j=1}^{m} \sum_{i=1}^{n} |a_{ij}|^2 \right]^{1/2}
$$

with $a_{ij}$ are the entries of the matrix $A$. 