February, 2025

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. Given a nonsingular $n \times n$ matrix A, and $n \times 1$ vectors $x, \hat{x}, b = Ax$, and $\hat{b} = A\hat{x}$.

   (a) Show the following well-known identity

   $$\frac{\|x - \hat{x}\|}{\|x\|} \frac{\|b\|}{\|b - \hat{b}\|} \leq \text{cond}(A)$$

   where $\text{cond}(A)$ is the condition number of matrix A.

   (b) Suppose that b is an eigenvector of A with eigenvalue λ, and $b - \hat{b}$ is an eigenvector of A with eigenvalue ω. Show that

   $$\frac{\|x - \hat{x}\|}{\|x\|} \frac{\|b\|}{\|b - \hat{b}\|} = \frac{|\lambda|}{|w|}$$

   in this special case.

   (c) If the eigenvalues of A are $100, -50, 3 \pm 5i$, then find a lower bound for $\text{cond}(A)$ using the result of Part (b) and under the assumptions of Part (b).

2. In iteratively solving the linear system $Ax = b$ ($\det A \neq 0$), we write $A = P - N$ with P nonsingular and generate a sequence $x^{(k)}$ by the formula

   $$Px^{(k+1)} = b + Nx^{(k)},$$

   starting with some initial guess $x^{(0)}$. Denote the residual $r^{(k)} = b - Ax^{(k)}$.

   (a) Show that the iteration formula can be equivalently expressed as

   $$x^{(k+1)} = x^{(k)} + P^{-1}r^{(k)}.$$

   (b) Let $w > 0$ be a constant. Define the Richardson method by the formula

   $$x^{(k+1)} = x^{(k)} + wP^{-1}r^{(k)}.$$

   Show that the method converges if $w|\lambda|^2 < 2\alpha$ for any eigenvalue $\lambda = \alpha + i\beta$ of $P^{-1}A$. Here, $|\lambda|$ denotes the norm (magnitude) of the complex number λ.

3. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

   (a) Find a singular value decomposition (SVD) of $\mathbf{A}$

   (b) Calculate the pseudo-inverse of $\mathbf{A}$

   (c) Find the least squares solution (LSS) to $\mathbf{Ax} = \mathbf{b}$

      i. by using QR–factorization

ii. by using singular value decomposition (SVD) and pseudo-inverse

4. Let $\mathbf{A} = \begin{bmatrix} 7.2 & 0.5 & -0.2 \\ -0.2 & 4.8 & -0.3 \\ -0.6 & 0.4 & -2.2 \end{bmatrix}$

(a) Use the Gershgorin theorem(s) to show that the matrix $\mathbf{A}$ is non-singular and has real eigenvalues.

(b) Does the power method converge if it is used to find one of the eigenvalues of the matrix $\mathbf{A}$? Explain your answer by making use of the Gershgorin theorem(s).

(c) Assume that we pick a shift $\mathbf{s} = 6.1$ and use inverse iteration; that is, apply power method iteration to $(\mathbf{A} - \mathbf{sI})^{-1}$. Can the Gershgorin theorem(s) be used to prove that the inverse iteration with shift $\mathbf{s}$ will converge? Explain your answer.