February, 2023

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours

1. (a) Fit a parabola of the form $y = a + bx^2$ to the following data by using the least squares approximation.

x	0	1	-2	3
y	1	1	-1	0

(b) Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}$. Find the singular value decomposition (SVD) of A and the rank of A.

(c) Consider the linear system $Ax = b$, for $b \in \mathbb{R}^3$ and A is the matrix given in part (b). State the condition such that the equation $Ax = b$ has a solution, and the condition such that the solution is unique.

(d) Find the pseudoinverse of the matrix A given in part (b).

(e) Find the solution of the system given in part (c) for $b = \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix}$.

2. (a) Prove the identity

$$A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$$

and hence deduce that

$$\|A^{-1} - B^{-1}\| \leq \|A^{-1}\| \, \|B - A\| \, \|B^{-1}\|$$

(b) Prove that if $B = A + \delta\, A$ where $\|\delta\, A\| \, \|B^{-1}\| = \delta < 1$, then it follows that

$$\|A^{-1}\| \leq \frac{1}{1-\delta} \|B^{-1}\|, \qquad \|A^{-1} - B^{-1}\| \leq \frac{\delta}{1-\delta} \|B^{-1}\|$$

(c) Prove that if $x = A^{-1}b$ and $x + \delta\, x = (A + \delta\, A)^{-1}b$, then

$$\|\delta\, x\| \leq \frac{\delta}{1-\delta} \|x + \delta\, x\| \quad \text{where} \quad \delta = \|\delta A\| \|B^{-1}\| < 1$$

and

$$\|\delta\, x\| \leq \frac{\epsilon}{1-\epsilon} \|x\| \quad \text{where} \quad \epsilon = \|\delta A\| \|A^{-1}\| < 1.$$

3. Let $A \in \mathbb{R}^{n\times n}$ be symmetric and positive definite matrix. Consider the following iteration

$$
\begin{aligned}
&\text{Choose} \quad A_0 = A \\
&\text{for} \quad k = 0, 1, 2, \ldots, \\
&\quad \text{Compute Cholesky factor } L_k \text{ of } A_k \text{ (so } A_k = L_k L_k^T) \\
&\quad \text{Set} \quad A_{k+1} = L_k^T L_k \\
&\text{end}
\end{aligned}
$$

where L_k is lower triangular with positive diagonal elements.

(a) Show that A_k is similar to A and that A_k is symmetric positive definite (thus the iteration is well-defined).

(b) Consider

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}, \quad a \geq c$$

For this matrix, perform one step of algorithm above and write down A_1.

(c) Use the result from (b) to argue that A_k converges to the diagonal matrix $D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda 2 \end{bmatrix}$, where the eigenvalues of A are ordered as $\lambda_1 \geq \lambda_2 > 0$.

4. Given the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & 3 \\ -2 & -6 \\ 1 & 0 \end{bmatrix}$$

(a) Find the reduced QR-factorization by applying Gram Schmidt ortogonalization to the columns of A.

(b) Find the full QR-factorization of A?