September, 2023

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Let $f(x)$ has the following values:

$$f(0) = 1, f'(0) = 1, f''(0) = 0, f(1) = 2, \text{ and } f'''(1) = 72$$

Find a polynomial $p(x)$ of minimal degree interpolating this data, i.e.,

$$f(0) = p(0), f(1) = p(1), f^{(j)}(0) = p^{(j)}(0) \text{ for } j = 1, 2, \text{ and } f^{(3)}(1) = p^{(3)}(1)$$

2. Suppose a value V is computed with a numerical procedure $\phi(h)$ and that

$$\lim_{h \to 0} \phi(h) = V$$

Assume there exists an asymptotic error expansion for $\phi(h)$ of the form

$$V - \phi(h) = c_1 h + c_2 h^2 + c_3 h^3 + \dots$$

(a) How should the values $\phi(h)$ and $\phi\left(\dfrac{h}{3}\right)$ be combined to yield an approximation to V which is $O(h^2)$?

(b) How should the values $\phi(h)$, $\phi\left(\dfrac{h}{2}\right)$, $\phi\left(\dfrac{h}{3}\right)$ be combined to yield an approximation to V which is $O(h^3)$?

3. The following parts are independent.

(a) Use the method of undetermined coefficients to construct a quadrature formula of the type

$$\int_0^1 f(x)\, dx = af(0) + bf(1) + cf''(w) + E(f)$$

having maximum degree of exactness k. Find a, b, c, w and k. ($E(f)$ denotes the error term).

(b) The following quadrature formula integrates polynomials of degree ≤ 2 exactly:

$$\int_0^h f(x)d\,x \approx \frac{3h}{4} f\left(\frac{h}{3}\right) + \frac{h}{4} f(h)$$

By using Peano Kernel theorem, derive an error bound of the form Ch^4 for this quadrature rule. Here C is a constant independent of h and assume $f \in C^3[0, h]$.

4. Suppose that the iterative method of the form

$$x_{n+1} = x_n - \frac{f(x_n)}{h(x_n)}$$

converges to a point α which is a root of the function $f(x)$, but not the root of the function $h(x)$. Find the relationship(s) between $f(x)$ and $h(x)$ such that the order of the convergence of the method is 3.