February, 2017

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Determine all values of a, b, c, d, e and f for which the following function $S(x)$ is a cubic spline

$$S(x) = \begin{cases} ax^2 + b(x-1)^3, & x \in (-\infty, 1], \\ cx^2 + d, & x \in [1, 2], \\ ex^2 + f(x-2)^3, & x \in [2, \infty). \end{cases}$$

2. (a) Derive a quadrature formula for

$$\int_{-1}^{1} x^2 f(x) \mathrm{d}x \approx \sum_{i=0}^{1} A_i f(x_i)$$

which is exact for polynomials of degree ≤ 3.

(b) Give an upper bound for the error made in the formula found in part (a).

(c) Evaluate $\displaystyle\int_0^2 (x-1)^2 e^x \mathrm{d}x$ by using the quadrature formula found in part (a).

3. (a) Use a suitable <u>*interpolating polynomial*</u> and its error term to derive the differentiation formula

$$f'(x_1) = \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{h^2}{6} f'''(\xi), \quad \xi \in (x_1 - h, x_1 + h)\,.$$

(b) Let $f(x) = e^{2x+1}$. Approximate $f'(1.4)$ by using the numerical differentiation formula obtained in part(a) and the values $f(1.3)$, $f(1.5)$, then approximate <u>the error</u>.

(c) The formula in part (a) can be written as

$$f'(x_1) = \frac{f(x_1+h) - f(x_1-h)}{2h} + K_1 h^2 + O(h^4)$$

where the constant $K_1 = -\dfrac{f'''(\xi)}{6}$. Use extrapolation to derive an $O(h^4)$ formula for $f'(x_1)$.

4. Given the polynomial $p(z) = z^4 + 2z^3 - 3z^2 + 2$.

   (a) Locate the roots of $p(z)$ in the complex plane.

   (b) Construct the synthetic division table for $p(z)$ for $z_0 = 2$; that is, write $p(z)$ as

$$p(z) = q_3(z)(z-2) + r_0,$$
$$q_3(z) = q_2(z)(z-2) + r_1$$
$$q_2(z) = q_1(z)(z-2) + r_2,$$
$$q_1(z) = q_0(z)(z-2) + r_3.$$

   (c) Write $p(z)$ in Taylor series expansion around $z_0 = 2$ using part (b).

   (d) If $x_0 = 2$ is an initial estimate to one of the real zeros of $p(z)$, carry out one iteration in Newton's method to approximate the root x_1 by using part (c).