March, 2021

Graduate Preliminary Examination
Numerical Analysis II
Duration: 3 Hours

1. Given the functional
$$L(f) = f(1) - 2f(0) + f(-1)$$
for all functions $f \in C[-1,1]$.

   (a) Show that $L(p_1(x)) = 0$ for all first-degree polynomials $p_1(x)$ on $[-1,1]$.

   (b) Show that
$$|L(f)| \leq \|L\| \cdot E_1(f) \quad \text{for all } f \in C[-1,1]$$
   where
$$\|L\| = \sup_{\|f\|=1} |L(f)|, \qquad E_1(f) = \inf \|f - p_1\|_\infty,$$
$$\|f - p_1\|_\infty = \max_{x\in[-1,1]} |f(x) - p_1(x)|.$$

   (c) Show that the curve $f(x) = e^x$ can not be approximated by a straight line in $[-1,1]$ with an error which is less than $\dfrac{e - 2 + e^{-1}}{4} \approx 0,271$.

2. Let f be a twice continuously differentiable function on $\mathbb{R}$ which has a unique root on the interval $[0,9]$. Suppose that for all $x \in [0,9]$, $f'(x) \geq 2$ and $|f''(x)| \leq 6$ are satisfied.

   (a) Determine if the secant iteration converge for any $x_0, x_1 \in [0,9]$.

   (b) If the secant iteration converge, find the number of iterations required to get an error that is less than 10^{-8}.

   (c) If the convergence of the secant method is not guaranteed for all $x \in [0,9]$, decide the number of steps of bisection needed before convergence will be guaranteed for the secant iteration. Explain your answer.

3. (a) Let $\phi(h)$ be an approximation for K for each $h > 0$ such that
$$K = \phi(h) + C_1 h + C_2 h^2 + C_3 h^3 + \dots$$
   with arbitrary constants C_1, C_2 and C_3. Use the values $\phi(h)$, $\phi(h/3)$, $\phi(h/9)$ to produce an approximation to K of order h^3 (i.e. $O(h^3)$).

   (b) Consider the difference formula
$$f'(x_0) = \frac{1}{f}(f(x_0 + h) - f(x_0)) - \frac{h}{2} f''(x_0) - \frac{h^2}{6} f'''(x_0) + O(h^3)$$
   for $f'(x)$ at x_0. Use this formula with the extrapolation of part (a) to derive an $O(h^3)$ formula for $f'(x_0)$.

4. Consider numerical integration

$$I_{app} = \sum_{j=0}^{n} \beta_j f(x_j), \quad x_j \in [-1, 1]$$

for the integral

$$I = \int_{-1}^{1} f(x)w(x)dx$$

where w is positive weight function in [-1,1]. Let

$$\Omega_{n+1}(x) = \prod_{j=1}^{n}(x - x_j)$$

denote the polynomial of degree $n + 1$ associated with the distinct quadrature nodes $x_0, x_1, x_2, \ldots, x_n$. Prove that

$$\int_{-1}^{1} \Omega_{n+1}(x)p(x)w(x)dx = 0$$

for any polynomial of degree less or equal to $m - 1$ if and only if the quadrature formula is exact for all polynomials of degree less or equal $n + m$.