Graduate Preliminary Examination
Ordinary Differential Equations
(3 hours)

September 20, 2010

1. Let $x(t, t_0, x_0)$ and $z(t, \bar{t}_0, \bar{x}_0)$ denote, respectively, solutions of
$$x' = f(t, x), \qquad x(t_0) = x_0$$
and
$$z' = f(t, z) + g(t, z), \qquad z(\bar{t}_0) = \bar{x}_0$$
where
(i) $f \in C(D)$, $|f(t, x)| \leq M$ for some $M > 0$, f is Lipschitz in x with Lipschitz contant L;
(ii) $g \in C(D)$, $|g(t, x)| \leq K$ for some $K > 0$.

   (a) Show that
$$|x(t) - z(t)| \leq \left(|x_0 - \bar{x}_0| + (M + K)|t_0 - \bar{t}_0| + \frac{M}{L}\right) e^{L|t-t_0|} - \frac{K}{L}.$$
   (b) Use part (a) with $g(t, x) \equiv 0$ to prove that the solution $x(t) = x(t, t_0, x_0)$ is continuous in (t_0, x_0) for t in a compact subset of real numbers.

2. Consider the systems
$$x' = Ax \qquad (1)$$
$$y' = Ay + f(t, y) \qquad (2)$$
where A is an $n \times n$ constant matrix and $f(t, y)$ is a continuous function defined on $R \times R^n$. Suppose that there exists a continuous function $\alpha(t)$ such that
$$||f(t, y)|| \leq \alpha(t)||y||, \qquad \int_a^\infty \alpha(t)\, dt < \infty, \quad (a \in R).$$
Show that if all solutions of (1) are bounded, then so are all solutions of (2). What would you say if A were not a constant matrix?

3. Consider the equation
$$(\frac{1}{1+t}x')' + (1 + \sin t)x = 0$$
   (a) Show that every solution of the equation has at least one zero in $[0, \pi]$.
   (b) Show that there is a solution having at least two zeros in $[0, \pi]$. Is it possible for any solution to have more than two zeros on $[0, \pi]$?.

4. Let
$$A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix}.$$
   (a) Find the fundamental matrix e^{At} of of the linear system
$$x' = Ax.$$
   (b) Use part (a) to show that all solutions are periodic. What is the common period of the solutions?
   (c) Show that the zero solution is stable. Is it asymptotically stable?