PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 2012/1

(1) Consider the differential equation $x'' + \omega^2 x = h(t)$, where $h(t)$ is continuous on $(t_1, t_2), \omega$ is a non-zero real constant. Show that the general solution is given by

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{1}{\omega}\int_{t_0}^{t} \sin\omega(t-s)h(s)ds,$$

where A and B are real constants and $t_0 \in (t_1, t_2)$ is a fixed real number. Use the preceding formula to find an integral equation that is equivalent to the nonlinear differential equation $x'' + \omega^2 x = f(t, x)$.

(2) Consider the linear differential equation

$$x' = (t^m A_0 + t^{m-1}A_1 + \ldots + A_m)x,$$

where $A_i, i = 0, 1, \ldots, m$, are constant n by n matrices, $x \in R^n$. Assume that the eigenvalues of A_0 have negative real parts. Prove that the solution $x = 0$ is asymptotically stable.

Hint: Introduce a new independent variable $s = (m+1)^{-1}t^{m+1}$.

(3) Consider the IVP

$$\begin{aligned} x_1' &= (-1 + \sin t)x_2 + \frac{x_1}{1 + x_2^2} + 5t \\ x_2' &= 2x_1 + (2 + \cos t)\frac{x_2}{1 + x_1^2} - t \\ x_1(0) &= 1, \;\; x_2(0) = 0. \end{aligned}$$

(a) Show that the IVP has a unique solution $x = x(t)$ defined on an interval $(-c, c)$ for some $c > 0$.

(b) Show that

$$\| x' \|_1 \leq 5 \| x \|_1 + 6t, \;\; t \geq 0.$$

Recall that $\| y \|_1 = |y_1| + |y_2|$.

(c) Use part (b) and the fact that $D^+ \| x \|_1 \leq \| x' \|_1$ to show that the solution is defined for all $t \geq 0$. Here D^+x is the upper Dini derivative of x.

(d) What can you say if $t \leq 0$?

(4) Let a be a continuous function satisfying $a(t + 2\pi) = a(t)$ for all $t \in \mathbb{R}$. Consider

$$x' = a(t)x.$$

Note that $x(t) = e^{\int_0^t a(s)ds}$ is a solution.

(a) Verify the e Floquet theorem.

(b) Calculate the Floquet exponent and the Floquet multiplier. Is there a periodic solution?

(c) Find the related constant coefficient equation.

(d) Answer (b) and (c) in the special case $a(t) = \sin t$.