PRELIMINARY EXAM PROBLEMS
Differential Equations (ODE), 3 hours, 2005/2

1. Consider the IVP

$$x_1' = -\frac{1}{t}x_2 + \sin t,$$

$$x_2' = \frac{1}{t}x_1 + \cos t,$$

$$x_1(1) = x_2(1) = 0.$$

Show that the IVP has a unique solution defined on $(0, \infty)$.

2. Let $p(t)$ be continuous function defined on $[1, \infty)$ such that

$$\int_1^\infty |p(t) - c| dt < \infty, \; c > 0.$$

(a). Show that all solutions of

$$y'' + p(t)y = 0 \tag{1}$$

are bounded on $[1, \infty)$. (Hint: rewrite the equation in the form $y'' + cy = (c - p(t))y$).
(b). What can you say about the stability of the zero solution?
(c). Show that all solutions of (1) need not to be bounded if $c = 0$. (Hint: $p(t) = \frac{a}{t^2}$).

3. Let $A(t)$ be a continuous matrix for all $t \in R$. Let $P(t)$ be the matrix solution of

$$X' = A(t)X.$$

Show that $P(t)P^{-1}(s) = P(t - s)$ for all $t, s \in R$, if and only if $A(t)$ is a constant matrix.

4. Consider the following scalar equation

$$x' = c(t)x, \tag{2}$$

where function $c(t) : R \to R$ is defined in the following way:

$$c(t) = \begin{cases} t, & \text{if} \quad 0 \le t \le \frac{1}{2}, \\ 1 - t, & \text{if} \quad \frac{1}{2} \le t \le 1 \end{cases}$$

and $c(t)$ is $1-$ periodic.
(a). Prove that (2) does not have a nontrivial 1-periodic solution.
(b). Does the equation have a nontrivial solution with another period?