Graduate Preliminary Examination
Ordinary Differential Equations
(3 hours)

February 16, 2011

1. Let $y_1 = y_1(x) = x^2$ and $y_2 = y_2(x) = x^5$.
   (a) Verify that y_1 and y_2 are linearly independent functions on $(0, \infty)$.
   (b) Find functions $h_1(x)$ and $h_2(x)$ such that $\{y_1, y_2\}$ is a fundamental set of solutions for the second order linear homogeneous equation
   $$y'' + h_1(x)y' + h_2(x)y = 0.$$
   (c) Solve the equation found in part (b) with the initial condition $y(1) = 1, y'(1) = -2$. Hint: Use the general solution.

2. Consider the following scalar differential equation
   $$x'(t) = x(\ln x)^2. \tag{1}$$
   (a) Determine the equilibrium, $x^* > 0$, of the equation (1).
   (b) Find the explicit solution $x(t, 0, x_0)$ and determine its domain.
   Consider $(a)\ \ 0 < x_0 < x^*, (b)\ \ x_0 = x^*, (c)\ \ x_0 > x^*$.
   (c) Can you investigate stability of the equilibrium , x^*, by applying the linearization technique? If not, why?
   (d) Can you investigate the stability by some another method? Show that the equilibrium is unstable.

3. Consider the system
   $$\frac{dz}{dt} = A(t)z + f(z) \tag{2}$$
   where $z \in R^n, A(t)$ is a continuous periodic, $n \times n$, matrix of period ω, $f(z)$ is continuous in some region about $z = 0$, and all Floquet multipliers of the system
   $$\frac{dy}{dt} = A(t)y \tag{$*$}$$
   are inside of the unit circle.

(a) The state transition matrix $\Phi(t,s) = \Psi(t)\Psi^{-1}(s)$ of (*) satisfies

$$||\Phi(t,s)|| < K e^{-\alpha(t-s)}, t \geq s,$$

where K and α are some positive numbers, and $\Psi(t)$ is a fundamental matrix of (*).

(b) If the inequality $||f(z)|| \leq L||z||$ is valid, and $KL < \alpha$, then the trivial solution, $z \equiv 0$, of (2) is asymptotically stable.

Hint: Apply the Floquet theory and Gronwall inequality.

4. (a) State the Sturm-Picone comparison theorem.

(b) Show that every solution of

$$x'' + \left(1 - \frac{1}{t}\right)x = 0$$

has a sequence of zeros $\{t_n\}$ that is unbounded from above.

(c) Show also that $\lim_{n\to\infty} |t_n - t_{n-1}| = \pi$.