PRELIMINARY EXAM PROBLEMS
Differential Equations (PDE), 3 hours, 13.09.2006

1. Let Ω denote the unbounded set $|x| > 1$. Function $u \in C^2(\bar{\Omega})$ satisfies $\Delta u = 0$ in Ω and $\lim_{x\to\infty} u(x) = 0$. Show that
$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|.$$
Hint: Apply the maximum principle to a spherical shell.

2. (a). Solve the following problem
$$xu_x + yu_y = u + 1, \quad u|_\Gamma = x^2$$
if $\Gamma = \{(x, y) : y = x^2\}$.

   (b). Use d'Alembert's formula to determine $u(1, 2)$ if
$$\begin{aligned} &u_{tt} - 4u_{xx} = 0, 0 < x < 2, t > 0, \\ &u(x, 0) = \sqrt{x}, \quad u_t(x, 0) = 2 - x, \quad 0 \le x \le 2, \\ &u(0, t) = 0, \quad u_t(2, t) = 0. \end{aligned}$$

3. Solve the following problem by Fourier's method.
$$\begin{aligned} &u_{tt} = u_{xx} + 2a, \quad 0 < x < l, \quad a - constant, \\ &u(0, t) = 0, \quad u(l, t) = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0. \end{aligned}$$

4. Consider the following initial value problem
$$\begin{aligned} u_t - u_{xx} = 0, \quad 0 < x < 1, \quad t > 0, \\ u(x, 0) = x, \quad 0 \le x \le 1, \\ u(0, t) = \sin t, \quad u(1, t) = \cos t, \quad t \ge 0. \end{aligned}$$
Using the maximum principle, show that:

   (a) $\quad u(x, t) \le 1$ for $t \in [0, T]$ $\quad$ for every $T > 0$;

   (b) the problem has at most one solution.