METU - Department of Mathematics
Graduate Preliminary Exam

Partial Differential Equations

Duration : 180 min. Fall 2008

1. Solve the following initial value problem.

$$u_x{}^2 - 3u_y{}^2 = u, \quad u(x,0) = x^2.$$

2. Let D be the region $(0,1) \times (0,1) \subset \mathbb{R}^2$ and let $u(x,y) \in C^2(D) \cap C^0(\overline{D})$ be a non-constant function. Set $M = \max(u)$ in $\overline{D}$.

   a) Show that if $u(x,y)$ solves the equation

$$\nabla^2 u(x,y) + a(x,y)u_x + b(x,y)u_y = F(x,y) \quad \text{in } D$$

   and if $F(x,y) > 0$ in D, then $u(x,y) < M$ for all $(x,y) \in D$.

   b) True or false ? The same conclusion holds if $u(x,y)$ solves the equation $u_{xy} = 0$ in D. (**Prove the statement or give a counter example**).

3. Consider the following Dirichlet problem.

$$\nabla^2 u = 0 \quad in\ \Omega = \{(r,\theta) : r > 1, 0 < \theta < \pi/2\}$$

$$u(r,0) = u(r,\pi/2) = 0 \text{ for } r \geq 1$$

$$u(1,\theta) = \sin(2\theta) \text{ for } 0 < \theta < \pi/2.$$

   a) Find the **bounded** solution of this problem.

   b) Find an unbounded solution, if there is any.

   c) Write a Neumann problem in Ω for which the function $u(r,\theta)$ of part (a) is a solution.

4. Let $G(x,t)$ be the heat kernel $G(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$.

a) Show that $u(x,t) = 2\int_0^t G(x,t-t')f(t')dt'$ satisfies

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in} \quad \{(x,t) : x > 0,\ t > 0\}$$

$$u(x,0) = 0,\ x > 0.$$

Hint : Do **not** verify the uniform convergence of the integral, but indicate when you use this property.

b) Verify that $-\frac{\partial u}{\partial x}|_{x=0} = f(t),\ t > 0.$

Hint : In the integral, first make the change of variable given by $s^2 = \frac{x^2}{4(t-t')}$.