METU - Department of Mathematics
Graduate Preliminary Exam

Partial Differential Equations

Duration : 180 min. Fall 2010
Each question is 25 pts.

1. Determine if the following Cauchy problem has a solution in the neighbourhood of the point (1,0)

$$yz_x - xz_y = 0,$$

   a) $z = 2y$ and $x = 1$

   b) $z = 2y$ and $x = 1 + y$.

2. Let $\Omega = \{x \in \mathbb{R}^3 : \quad |x| > 1\}$. Let $u \in C^2(\bar{\Omega})$ and suppose that u satisfies the Laplace equation in Ω

$$\Delta u(x) = 0, \text{for all } x \in \Omega.$$

   Show that

$$\max_{\bar{\Omega}} |u| = \max_{\partial\Omega} |u|$$

   if $\lim_{x \to \infty} u(x) = 0$.

3. For the wave equation in $\mathbb{R}^3$

$$u_{tt} = u_{xx} + u_{yy} + u_{zz}$$

   find a general form of the plane wave solution. That is, a solution of the form $u(t, x, y, z) = v(t, s)$ where $s = \alpha x + \beta y + \gamma z$ and α, β, $\gamma \in \mathbb{R}$, $\alpha^2 + \beta^2 + \gamma^2 = 1$.

4. Consider the P.D.E.

$$\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = xe^{-y}. \quad (*)$$

a) Determine the type of this PDE and find the characteristic curves.

b) Consider the given PDE as a first order PDE for $q = \dfrac{\partial z}{\partial y}$.

Solve this first order PDE for q. Then find the general solution of (*).

c) Find two different solutions $z(x, y)$ which satisfy the condition

$$z(x, x) = 1 \text{ for } x \in \mathbb{R}.$$

d) True or false ? Explain.

There exists a solution $z(x, y)$ such that $z(x, x) = 1$, $\dfrac{\partial z}{\partial \mathbf{n}} = 0$ where $\mathbf{n}$ is the unit normal vector to the line $y = x$ in $\mathbb{R}^2$.