PRELIMINARY EXAM - Sep. 2012

Partial Differential Equations

Q.1	Q.2	Q.3	Q.4	Total

Duration : 3 hr.

Each question is 25 pt.

1. Consider the PDE

$$x^4(\frac{\partial z}{\partial x})^2 - yz\frac{\partial z}{\partial y} - z^2 = 0 \text{ in } \Omega = \{(x, y, z) \in \mathbf{R}^3 : x > 0,\ y > 0,\ z > 0\} \quad (*).$$

a) Transform the given PDE by using the change of variables

$$X = \frac{1}{x},\ Y = \frac{1}{y},\ Z = \ln(z).$$

b) Solve the PDE you obtained in (a) to get the **complete integral** of the original PDE (*).

c) Find all singular solutions, if any, of (*).

2. Consider the problem

$$u_{xx} - u_t = tu \text{ for } (x, t) \in \mathbb{R} \times (0, \infty)$$
$$u(x, 0) = \phi(x).$$

a) Determine the ODE satisfied by the function $f(t)$ if $u(x, t) = f(t)v(x, t)$ where $v(x, t)$ is the solution of the problem

$$v_{xx} - v_t = 0 \text{ for } (x, t) \in \mathbb{R} \times (0, \infty)$$
$$v(x, 0) = \phi(x).$$

b) Determine $f(t)$.

c) Let $u_1(x,t)$, $u_2(x,t)$ be the solutions of the given problem corresponding to the boundary conditions $\phi_1(x)$, $\phi_2(x)$ respectively. Show that if

$$\sup_{\mathbb{R}} |\phi_1(x) - \phi_2(x)| \leq 1$$

then $|u_1(x,t) - u_2(x,t)| \leq e^{-t^2/2}$ for all $(x,t) \in \mathbb{R} \times (0,\infty)$.

3. Consider the equation

$$U_{xy} + 2y\, U_{xx} + U_x = 0\ .$$

(A) Determine the type of this equation.

(B) Introduce a suitable system of coordinate in which the principal part of this equation assumes a canonical form.

(C) Find the general solution of the above equation.

(D) Find the solution of the above equation which satisfies

$$U_x(x,0) = x^2 \quad \text{and} \quad U(0,y) = 0\ .$$

(Hint : Notice that $M = x - H(y)$ is a solution of the equation

$$H'(y)\, M_x + M_y = 0).$$

4. Let $\Omega \subset \mathbb{R}^2$ be an open connected and bounded region and let $u : \overline{\Omega} \to \mathbb{R}$ be a continuous functon which is C^2 on Ω. Show that if u satisfies

$$\nabla^2 u = u^3 \text{ in } \Omega$$

$$u|_{\partial\Omega} = 0$$

then $u \equiv 0$ in Ω.