Partial Differential Equations

Problem 1. For the equation

$$xz_x + yz_y = z + 1$$

give an initial curve such that:
(a) there exists infinitely many solutions passing through the initial curve;
(b) there exists no solution passing through the initial curve.

Problem 2. Find a general form of a radially symmetric solution $u = u(r,t)$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, for the wave equation

$$u_{tt} = c^2 \Delta u \qquad x \in \mathbb{R}^3,\ t \in \mathbb{R}.$$

Problem 3. Let Ω be an open bounded domain in $\mathbb{R}^2$. Suppose that $u \in C^2(\Omega)$ satisfies

$$u_{xx} + u_{yy} + xu_x + yu_y = 1, \qquad (x,y) \in \Omega$$

$$u(x,y) = 0 \qquad (x,y) \in \partial\Omega.$$

Show that $u(x,y) \leq 0$ in Ω.

Problem 4. For the following problem

$$\begin{array}{ll} u_{tt} + k^2 u_t = \Delta u, & (x,t) \in Q_T \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x) & x \in D \\ u(x,t) = 0, & (x,t) \in S_T \end{array}$$

where D is an open bounded domain in $\mathbb{R}^3$ and $Q_T = \{(x,t) : x \in D,\ 0 < t \leq T\}$, $S_T = \{(x,t) : x \in \partial D,\ 0 \leq t \leq T\}$. Show that:
(a)

$$E(t) = \int_D \left[\frac{1}{2}u_t^2 + \frac{1}{2}|\nabla u|^2\right] dx$$

is non-increasing function;
(b) the problem has at most one solution.