19 February 2004

Graduate Preliminary Examination
Partial Differential Equations
Duration: 3 hours

1. Find the characteristics of the equation $u_x^2 + u_y^2 = u^2$ and determine the integral surface passing through $x^2 + y^2 = 1$, $u = 1$.

2. Find the solution $u = f(x^2 - c^2t^2) = f(s)$, where $f(0) = 1$, of

$$u_{tt} - c^2 u_{xx} = \lambda^2 u, \quad \lambda : \text{ constant}$$

in the form of a power series.

3. a) Show that the problem

$$\frac{\partial}{\partial x}(e^x \frac{\partial u}{\partial x}) + \frac{\partial}{\partial y}(e^y \frac{\partial u}{\partial y}) = 0 \qquad \text{in } D$$

$$u = x^2 \qquad \text{on } \partial D$$

cannot have more than one solution if $u \in C(\overline{D}) \cap C^2(D)$ where

$$D := \{(x, y), x^2 + y^2 < 1\}$$

b) Can you extend the result in (a) to the problem

$$\begin{aligned} Lu &= -F(x, y) \quad \text{in } D \\ u &= f(x, y) \quad \text{on } \partial D \end{aligned}$$

where

$$Lu = \frac{\partial}{\partial x}[A(x, y)\frac{\partial u}{\partial x} + B_1(x, y)\frac{\partial u}{\partial y}] + \frac{\partial}{\partial y}[B_2(x, y)\frac{\partial u}{\partial x} + C(x, y)\frac{\partial u}{\partial u}\partial y]$$

is an elliptic operator.

4. Find the solution of the initial value problem

$$u_t + u = \Delta u, \quad x \in \mathbb{R}^n, \ t > 0$$

$$u(x, 0) = h(x), \quad x \in \mathbb{R}^n$$

where h is continuous and bounded in $\mathbb{R}^n$ and Δ is the Laplace operator in $\mathbb{R}^n$.