

PRELIMINARY EXAM PROBLEMS

Differential Equations (PDE), 2005/2, 3 hours

1. Solve the Dirichlet problem

$$u_{xx} + u_{yy} = 0, \quad x^2 + y^2 < 1,$$
$$u = y^4, \quad x^2 + y^2 = 1.$$

2. (a). Find, for all positive and negative values of a constant λ , the real solutions of the equation

$$\frac{\partial^2 z}{\partial x^2} = c^2 \frac{\partial z}{\partial t}$$

that are of the form $z = e^{\lambda t} \phi(x)$.

(b). If c is not an integer multiple of π , show that there exists a solution of this equation which remains finite as $t \rightarrow \infty$, which is zero when $x = 0$, and which assumes the value e^{-t} when $x = 1$. Find this solution.

3. (a). Let $u(x, t)$ be a solution of the equation

$$u_t - ku_{xx} = F(x, t), \quad k > 0, \tag{1}$$

for $\{(x, t) \mid 0 < x < L, t > 0\}$, where L is a fixed positive number, and $u(x, t)$ is continuous in $\{(x, t) \mid 0 \leq x \leq L, t \geq 0\}$. Prove that the maximum of $u(x, t)$ is attained at $t = 0$, or $x = 0$, or $x = L$, if $F(x, t)$ is negative valued in $\{(x, t) \mid 0 < x < L, t > 0\}$.

(b). Construct a counter example if $F(x, t)$ in (1) is positive in the region.

4. The function $\frac{-1}{2\pi} K_0(\alpha r)$ is a fundamental solution for the equation

$$\nabla^2 u - \alpha^2 u = 0 \quad \text{in } \Omega,$$

where α is a constant, $\Omega \subset R^2$ and $K_0(\alpha r)$ is the zero order modified Bessel function of the second kind, r is the distance from a fixed point (ξ, η) to any point (x, y) in Ω .

Prove that the Green's function for the equation above defined by

$$\nabla^2 G - \alpha^2 G = \delta(x - \xi)\delta(y - \eta) \quad \text{in } \Omega,$$

$$G = 0 \quad \text{on } \partial\Omega,$$

is unique, where δ and ∇^2 denote the Dirac Delta and Laplace operators respectively.