M.E.T.U.
DEPARTMENT OF MATHEMATICS

Preliminary Exam - Feb. 2013
Partial Differential Equations

Duration : 3 hr.

Each question is 25 pt.

1. Consider the following problem

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xy$$

$$u(1, y) = e^y.$$

a) Find the solution.

b) Discuss the existence and uniqueness of the solution in a neighbourhood of $(1, y_0)$ as y_0 varies.

2. Consider the following problem.

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in } \Omega = \{(x,t) : 2 > x > 0, t > 0\}$$

$$u(0,t) = u(2,t) = 0, \; u(x,0) = 1 - |x-1| \text{ for } 2 \geq x \geq 0.$$

a) Solve this problem by the method of separation of variables.

(Write the integral expressions for the coefficients, but do not compute the integrals).

b) Using the heat kernel, write the integral form of the solution of the problem

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \quad \text{in } \Omega = \{(x,t) : x \in \mathbb{R}, \; t > 0\}$$

$$u(x,0) = \begin{cases} 1 - |x-1| & \text{if } 2 \geq x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Show that $|u(x,t)| \leq 1$ in $\mathbb{R} \times (0, \infty)$.

c) Can you obtain the solution in (a) by restricting the solution in (b) to $\Omega = \{(x,t) : 2 > x > 0,\ t > 0\}$?

3. Consider the following Monge-Ampere equation in two variables

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - (\frac{\partial^2 u}{\partial y \partial x})^2 = 1.$$

a) Determine the general solution $u(x,y)$ which satisfies

$$\frac{\partial^2 u}{\partial x^2} = 1 = \frac{\partial^2 u}{\partial y^2}.$$

b) Find two distinct solutions of the BVP : $u(x,y)|_{x^2+y^2=1} = 1$.

(Hint : Replace 1 by -1 in (a) and find the corresponding general solution).

c) Verify that the mean value property

$$u(x_0,y_0) = \frac{1}{2\pi r}\int_{S^1} u(x,y)ds$$

(where S^1 is the circle $(x-x_0)^2 + (y-y_0)^2 = r^2$)

and the maximum principle do not hold for the solutions of the given equation.

4. Let $\Omega \subset \mathbb{R}^2$ be an open connected and bounded region, $\overline{\Omega}$ be the closure of Ω and $\partial\overline{\Omega}$ be the boundary. Let $f(x,y) \in C^2(\overline{\Omega})$ be a **subharmonic** function, that is f satisfies the inequality $\nabla^2 f \geq 0$ in Ω. Equivalently, for any $p \in \Omega$ and any disc $\overline{D}(p;r) \subset \overline{\Omega}$ one has

$$f(p) \leq \frac{1}{2\pi r}\int_{\partial D(p;r)} f ds.$$

a) Show that if $f(p) = \max_{\overline{\Omega}}(f)$ for some $p \in \Omega$, then f is constant.

b) Show that if $u(x,y)$ is harmonic in $\overline{\Omega}$ then $f(x,y) = |\nabla u(x,y)|^2$ is subharmonic.

c) Find a nonconstant subharmonic function f in $\overline{D}(0;1)$ such that $\max_{\overline{D}}(f) = 1$.

(Hint : You may use (b)).