Partial Differential Equations

Problem 1. For the given equation

$$xyz_x + xzz_y = yx$$

find a curve s.t. there exists infinitely many solutions passing through this curve.

Problem 2. Using Green's first identity

$$\int_{\partial D} v\,\frac{\partial u}{\partial n}\,dS = \iint_D \nabla v \cdot \nabla u\,dxdy + \iint_D v\Delta u\,dxdy$$

show that the Dirichlet problem

$$\begin{array}{ll} \Delta u + ku = q(x,y), & \text{in} \quad D \\ u = f(x,y) & \text{on} \quad \partial D \end{array}$$

has a unique solution if $k < 0$.

Problem 3. For equation

$$\Delta u(x) + u(x) = 0 \quad x \in \mathbb{R}^3$$

find the spherically symmetric solution that is solution of the form $u = f(r)$, where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$.

Problem 4. Let Ω is be a bounded domain in $\mathbb{R}^2$. Suppose that $u \in C^2(\Omega)$ satisfies

$$-\Delta u + \nabla u + e^{xy}u > 0 \quad \text{on} \quad \Omega,$$

and u assumes a minimum value on Ω at some interior point $x_0 \in \Omega$. Prove that $u(x_0) \geq 0$.