GRADUATE PRELIMINARY EXAMINATION
ANALYSIS I (REAL ANALYSIS)
Fall 2005
September 12^{th}, 2005

Duration: 3 hours

1. Let (X, S, μ) be a measure space, T be a metric space. Let $f : X \times T \to \mathbf{R}$ be a function. Assume that $f(\cdot, t)$ is measurable for each $t \in T$ and $f(x, \cdot)$ is continuous for each $x \in X$. Prove that if there exists an integrable function g such that for each $t \in T$, $|f(x,t)| \leq g(x)$ for $a.a.x$, then $F : T \to \mathbf{R}$, $F(t) = \int f(x,t)d\mu(x)$ is continuous.

2. Let $\mathcal{G}$ be a set of half-open intervals in $\mathbf{R}$. Prove that $\cup_{G \in \mathcal{G}} G$ is Lebesgue measurable.

3. a) Let $f_n = \sin n^2 x \in L_p[0,1]$, where $1 \leq p < \infty$. Show that $f_n \to 0$ weakly, but $f_n \not\to 0$ in measure.

b) Let $g_n = n^2 \chi_{[0, \frac{1}{n}]} \in L_p[0,1]$, where $1 \leq p < \infty$. Show that $g_n \to 0$ in measure, but $g_n \not\to 0$ weakly.

c) Let A_n be a measurable subset of $[0,1]$ for each $n, \chi_{A_n} \in L_1$, and $\chi_{A_n} \to f$ weakly in L_1. Show that f is not necessarily a characteristic function of some measurable set.

4. Let $f : \mathbf{R} \to \mathbf{R}$. If $f \in L_1(m) \cap L_2(m)$ where m denotes the Lebesgue measure, prove that

a) $f \in L_p(m) \;\; \forall \;\; 1 \leq p \leq 2$

b) $\lim\limits_{p \to 1^+} ||f||_p = ||f||_1$.