TMS

September 2011

Real Analysis

I. a) Let A_n be a sequence of measurable sets with $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Prove that $\mu(\overline{lim} A_n) = 0$

Hint: $\overline{lim} A_n = \bigcap_{k=1}^{\infty} \cup_{n \geq k} A_n$

b) Let $f \in L_p(\mu)$ and $\epsilon > 0$. Show that

$$\mu(\{x \in X : |f(x)| \geq \epsilon\}) \quad \leq \quad \epsilon^{-p} \int |f|^p d\mu$$

II. a) Show that $f(x) = \dfrac{1}{\sqrt{x}}$ is Lebebsgue integrable over $[0, 1]$.

b) Compute $\lim_n \int_0^1 \frac{n \sin x}{1+n^2\sqrt{x}} dx$ and justify your calculations.

III. Assume $\mu(X) < \infty$. If f_n is a sequence of measurable functions on X such that $f_n \to f$ a.e. then prove that $f_n \to f$ [meas] also holds.

State the theorem(s) you used.

IV. Assume that $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are two continuous functions such that $f(x) \leq g(x)$ holds for all $x \in [a, b]$. Set $A = \{(x, y) \in \mathbb{R}^2 : x \in [a, b]$ and $f(x) \leq y \leq g(x)\}$.

a) Show that A is a closed set (and hence a measurable subset of $\mathbb{R}^2$)

b) If $h : A \to \mathbb{R}$ is a continuous function, then show that h is Lebesque integrable over A and that

$$\int_A h d\lambda = \int_a^b (\int_{f(x)}^{g(x)} h(x, y) dy) dx$$