

METU MATHEMATICS DEPARTMENT
REAL ANALYSIS
SEPTEMBER 2015 - TMS EXAM

1. Formulate the Egoroff theorem (= the third Littlewood principle) and show that it fails in every measure space with infinite σ -finite measure.

Hint: Consider $f_n = \chi_{[n, n+1]}$

2. Let (X, \mathcal{A}, μ) be a measure space, and suppose $X = \bigcup_n X_n$, where $\{X_n\}_{n=1}^\infty$ is a pairwise disjoint collection of measurable subsets of X . Use the monotone convergence theorem and the linearity of the integral to prove that, if f is a non-negative measurable real-valued function on X ,

$$\int_X f d\mu = \sum_n \int_{X_n} f d\mu$$

Hint: Let $f_n = \sum_{k=1}^n f \chi_{X_k} = f \chi_{\bigcup_{k=1}^n X_k}$

3. Evaluate $\lim_{k \rightarrow \infty} \sum_{n=1}^{\infty} e^{-kn^2}$ and prove your answer by using a measure theory theorem.

Hint: Let $f_k : \mathbb{N} \rightarrow [0, \infty)$ be defined by $f_k(n) = e^{-kn^2}$, $n \in \mathbb{N}$.

4. Using the Fubini/Tonelli theorems to justify all steps, evaluate the integral

$$\int_0^1 \int_y^1 x^{-3/2} \cos\left(\frac{\pi y}{2x}\right) dx dy$$

Hint: Consider $\int \int |x^{-3/2} \cos(\frac{\pi y}{2x})| dy dx$ and apply to Tonelli's theorem.