M E T U Department of Mathematics

Graduate Preliminary Exam, Analysis	Fall 2020-2021	04.11.2020	10:00
Last Name : Name : Student No :	Signature :		
3 QUESTIONS			TOTAL 100 POINTS
1 | 2 | 3 | 4 | |	Duration: 180 minutes		

(1) (10+15 points) Let $\mathbf{m}$ denote the Lebesgue measure on $\mathbb{R}$.

a) Let $E \subseteq \mathbb{R}$ be non-Lebesgue measurable. Prove that $\mathbf{m}\left(\overline{E}\right) > 0$ where $\overline{E}$ denotes the closure of E.

b) Let $A \subseteq [0,1]$ be Lebesgue measurable and $\mathbf{m}(A) = c > 0$. Prove that for every $0 < \alpha < c$ there exists a *closed* set $B_\alpha \subseteq A$ such that $\mathbf{m}(B_\alpha) = \alpha$.
(**Hint.** First argue that the map $f : [0,1] \to [0,1]$ given by $f(x) = \mathbf{m}(K \cap [0,x])$ is continuous for every measurable $K \subseteq [0,1]$. Then try to use this fact together with approximation theorems regarding Lebesgue measurable sets.)

(2) (10+15+10+10 points) a) State Lebesgue's dominated convergence theorem.

In the remaining parts of this question, you will consider the measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \nu)$ with the measure ν given by

$$\nu(S) = \sum_{n \in S} \frac{1}{2^n}$$

b) Let $f : \mathbb{N} \to \mathbb{R}$ be a bounded function. Prove that f is integrable and

$$\int_{\mathbb{N}} f \; d\nu = \sum_{k=0}^{\infty} \frac{f(k)}{2^k}$$

(**Hint.** Set $f_n(x) = f(x) \cdot \chi_{\{0,1,2,\ldots n\}}(x)$. Observe that $f_n \longrightarrow f$ pointwise. Then apply an appropriate theorem to get the result.)

c) Let $f : \mathbb{N} \to \mathbb{R}$ be a function and $(f_n)_{n \in \mathbb{N}}$ be a sequence of functions from $\mathbb{N}$ to $\mathbb{R}$. Show that if $f_n \to f$ in measure, then $f_n \to f$ pointwise.

d) Let $g(x,k) = x^k$. Compute the integral

$$\int_{[1,3/2] \times \mathbb{N}} g(x,k) \; d(\mathbf{m} \times \nu)$$

in the product measure space $(\mathbb{R} \times \mathbb{N}, \mathcal{B}(\mathbb{R}) \otimes \mathcal{P}(\mathbb{N}), \mathbf{m} \times \nu)$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra of $\mathbb{R}$ and $\mathbf{m}$ is the Lebesgue measure. Explain each step of your computation by referring to the relevant theorems.

(3) (10+10+10 points) Let μ be the measure defined on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ given by

$$\mu(E) = \begin{cases} +\infty & \text{if } E \text{ is uncountable} \\ 0 & \text{if } E \text{ is countable} \end{cases}$$

and let $\mathbf{m}$ denote the usual Lebesgue measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$.

a) Show that $\mathbf{m} \ll \mu$, that is, $\mathbf{m}$ is absolutely continuous with respect to μ.

b) Show that there exists no Borel measurable function $f : \mathbb{R} \to [0, +\infty)$ such that

$$\mathbf{m}(E) = \int_E f d\mu$$

c) Explain why (a) and (b) together do not contradict the Radon-Nikodym theorem.