TMS

Spring 2010

Real Analysis

1. a) Show that $f(x) = \dfrac{\ln x}{x^2}$ is Lebesgue integrable over $[1, \infty)$ and $\int f d\mu = 1$

   b) A set E in $\mathbb{R}$ is said to be **null** if for any $\epsilon > 0$ we can cover E with countably many open intervals the sum of whose lengths is less than ϵ, i.e., $E \subset \cup_{n=1}^{\infty}(a_n, b_n)$ and $\sum_1^{\infty}(b_n - a_n) < \epsilon$.

   Show that any countable set in $\mathbb{R}$ is **null**.

2. Using Lebesgue Dominated Covergence Theorem, compute

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} e^{-kn^2}$$

   Hint: Consider $\mathbb{N}$ with the counting measure. Let

$$f_k : \mathbb{N} \to [0, \infty) \quad \text{be defined as} \quad f_k(n) = e^{-kn^2}. \quad \text{Use LDCT.}$$

3. a) Suppose $(f_n) \to f$ in measure and $(g_n) \to g$ in measure. Show $(f_n + g_n) \to f + g$ in measure.

   b) Let (f_n), (g_n) be sequences of measurable functions such that $(f_n) \to f$ in measure, $(g_n) \to g$ in measure and $f_n = g_n$ a.e. for every n. Show that $f = g$ a.e.

4. State Egoroff's theorem. Prove that in Egoroff's theorem the hypothesis $\mu(X) < \infty$ can be replaced by $|f_n| \leq g$ for all n where $g \in L^1(\mu)$