Preliminary Exam - February, 2015

Real Analysis

1) Let (X, M, μ) be a measure space, $g : X \to: [0, \infty]$ be a non-negative μ- measurable function. For each $E \in M$ define $\nu(E) = \int g\chi_E d\mu$.

a) Show that ν is a measure on (X, M).

b) Show that if f is any non-negative μ- measurable function then $\int f d\nu = \int fg d\mu$.

2) Let (X, M, μ) be a finite measure space, E_k be a sequence of sets in μ such that $\mu(E_k) > 1/100 \quad \forall k$. Let F be the set of points $x \in X$ which belong to infinitely many of these sets,E_k.

a) Show that $E \in M$.

b) Show that $\mu(E) \geq 1/100$

c) Show that conclusion (b) may fail if $\mu(X) = \infty$.

3) a) State the Dominated Convergence Theorem.

b) Let μ be a measure on the Borel subsets of $\mathbb{R}$, and $f \in L^1(\mu)$. Prove that the function $F(x) = \displaystyle\int_{(-\infty,x]} f d\mu$ is continous from the left.

c) Show that if $x \in \mathbb{R}$ and $\mu(x) = 0$ then F is continuous from the right at x.

4) Let μ, ν be finite measures on (X, M) and $\nu = \nu_1 + \nu_2$ be the Jordan decomposition of ν so that $\nu_1 \perp \mu$ and $\nu_2 << \mu$. Let $\lambda = \nu + \mu$.

a) Show that if A,B is a Hahn Decomposition for ν_1, μ then it is also a Hahn Decomposition for ν_1, ν_2.

b) Show that $\nu << \lambda$

c) Let $f = \frac{d\nu}{d\lambda}$. Show that $0 \leq f \leq 1 \quad \lambda - a.e.$ and the two sets $f^{-1}(\{1\}), f^{-1}([0, 1))$ form a Hahn Decomposition for ν_1, μ.

5) Let $f : \mathbb{R} \to \mathbb{R}$ be such that $\int_{-\infty}^{\infty} f dx$ converges in the usual Riemann sense, let m denote the Lebesgue measure on $\mathbb{R}$.

a) Show that if $f(x) \geq 0$ m-a.e then $f \in L^1(m)$.

b) Give an example showing that the non-negativity assumption in part (a) is necessary.