

Preliminary Exam - February, 2017

Real Analysis

- 1) Let  $(X, \mathcal{M}, \mu)$  be a measure space. Define  $\mu^* : \mathcal{P}(X) \rightarrow [0, \infty]$  by  $\mu^*(A) = \inf\{\mu(E) : E \in \mathcal{M}, A \subset E\}$ . Prove that
- $\mu^*$  is an outer measure on  $X$ .
  - $\forall A \in \mathcal{P}(X) \exists E \in \mathcal{M}$  such that  $A \subset E$  and  $\mu^*(A) = \mu(E)$ .
- 2) Suppose that  $\{f_n\}$  is a sequence of Lebesgue measurable functions on  $[0, 1]$  such that  $\lim_{n \rightarrow \infty} \int_0^1 |f_n| dm = 0$  and there is an integrable function  $g$  on  $[0, 1]$  such that  $|f_n|^2 \leq g$ , for each  $n$ .
- Prove that  $\lim_{n \rightarrow \infty} \int_0^1 |f_n|^2 dm = 0$
  - Prove that if  $\lim_n f_n = f$  exists a.e. then  $f$  integrable on  $[0, 1]$  and  $\int f dm = 0$
- 3) If  $f$  is a complex valued measurable function on  $(X, \mathcal{M}, \mu)$ , define

$$R_f = \{z : \mu(\{x : |f(x) - z| < \epsilon\}) > 0 \forall \epsilon > 0\}$$

Show that

- $R_f$  is closed.
  - If  $f \in L^\infty$  then  $R_f$  is compact.
- 4) Let  $(X, \mathcal{M}, \mu)$  be an arbitrary measure space and define  $\nu$  on  $\mathcal{M}$  by  $\nu(A) = 0$  if  $\mu(A) = 0$ ; and  $\nu(A) = \infty$  if  $\mu(A) > 0$ .
- Show that  $\nu$  is a measure on  $X$  and  $\nu \ll \mu$ .
  - Find  $\frac{d\nu}{d\mu}$ .