Preliminary Exam - February, 2018

Real Analysis

1) a) Let $f : X \to \mathbb{R}$ be integrable, where (X, μ) is a measure space. Show that the set $S = \{x : \quad f(x) \neq 0\}$ is of σ-finite measure.

b) Evaluate $\lim\limits_{k\to\infty} \sum\limits_{n=1}^{\infty} e^{-n^2/k}$

Justify your answer using measure theory theorems.

2) a) Let $f \in L_p[0,1]$ and let $\epsilon > 0$. Show that if $p \geq 1$ then

$m(\{x \in [0,1] : |f(x)| \geq \epsilon\}) \leq \epsilon^{-p} \int |f|^p dm$

b) Let $f_n \in L_p[0,1] \forall n$. Show that if

$\lim\limits_{n\to\infty} ||f_n - f||_p = 0$ holds in $L_p[0,1]$ then (f_n) converges in measure to f.

3) a) Show that if $f \in L_2 \cap L_4$ then $f \in L_3$ holds.

b) Let $1 \leq p < q < \infty$ Show $L_q[0,1] \subset L_p[0,1]$ holds.

4) a)State the Fubini Theorem

b) Using $f(x,y) = \begin{cases} 2 - 2^{-x} & \text{if } x = y \\ -2 + 2^{-x} & \text{if } x = y + 1 \\ 0 & \text{otherwise} \end{cases}$

where $f : X \times Y \to \mathbb{R}$, $X = Y = \mathbb{N}$ both equipped with counting measure, show that integrability of f in Fubini's Theorem cannot be removed from the hypothesis.