Real Analysis

Feb. 2019

I Let $A \subset [0,1]$ be a non-measurable set.

Let $B = \{(x,0) \in \mathbb{R}^2 : x \in A\}$

(13 pts) (a) Is B a Lebesgue measurable subset of $\mathbb{R}^2$?

(12 pts) (b) Can B be a closed subset of $\mathbb{R}^2$ for some such A?

II Let $f : [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \text{ is irrational} \\ 0 & \text{otherwise} \end{cases}$$

(8 pts) a) Show that f is measurable.

(8 pts) b) Is f Lebesgue integrable? if yes, find its Lebesgue integral.

(9 pts) c) Is f Riemann integrable? if yes, find its Riemann integral.

III a) If $f \in L^1(0,1)$, find $\displaystyle\lim_{k\to\infty} \int_0^1 k \ln\left(1 + \frac{|f(x)|^2}{k^2}\right) dx$

(13 pts)

<u>Hint:</u> Show $\ln(1+t) \le 2\sqrt{t}$ for $t \geq 0$.

(12 pts) b) Evaluate $\displaystyle\sum_{n=0}^{\infty} \int_0^{\pi/2} (1 - \sqrt{\sin x})^n \cos x \, dx$. Justify your steps.

IV a) Prove that $\displaystyle\int_0^{\pi/2} \sqrt{x \sin x}\, dx \le \frac{\pi}{2\sqrt{2}}$

(20 pts)

(5 pts) b) Prove that the inequality is strict.