Real Analysis Preliminary Exam
Feb 2024

1. (25 pts) Prove that the set

$$E = \{x \in [0,1] : \forall q \in \mathbb{N}, \exists p \in \mathbb{N} \ s.t \left| x - \frac{p}{q} \right| \leq \frac{1}{q^2}\}$$

   is Lebesgue measurable and its Lebesque measure is 0.

2. (10+10+5 pts) Let m be the Lebesgue measure on $[0,1]$, and μ is the counting measure on $[0,1]$.
   (a) Is m absolutely continuous with respect to μ?
   (b) Does the Radon-Nikodym derivative $dm/d\mu$ exist?
   (c) Does your result in part (b) contradict to Radon-Nikodym Theorem?

3. (25pts) Let $f \in L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Show that $\int_{\mathbb{R}} |f(x-y)g(y)|dy < \infty$ for a.e x.

4. (25pts) Show that if $\{f_n\} \in L^+$, f_n decreases pointwise to f and $\int f_k < \infty$ for some k then $\int f = \lim \int f_n$.