# M E T U Department of Mathematics

| Graduate Preliminary Exam, Analysis | Spring 2023 | 03 March 2023 | 10:00 |
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| Last Name :<br>Name :<br>Student No : | Signature : | | |
| 4 QUESTIONS | | TOTAL 100 POINTS | |
| 1 \| 2 \| 3 \| 4 | Duration: 180 minutes | | |

(1) (15+15 points) Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of non-negative real valued measurable functions and $f$ be a real valued measurable function on $(X, \mathcal{M}; \mu)$.

a) State Fatou's lemma.

b) Prove that if $f_n \to f$ in measure, then $\int_X f \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu$.

(2) (20 points) Consider the measures $\mu, \nu$ on $\mathcal{B}(\mathbb{R})$ given by

$$\mu(A) = \int_A \frac{1}{1+x^2} \, d\mathrm{m} \quad \text{and} \quad \nu(A) = \int_A e^{-x^2} \, d\mathrm{m}$$

where $\mathcal{B}(\mathbb{R})$ denotes the Borel $\sigma$-algebra of $\mathbb{R}$ and $\mathrm{m}$ denotes the Lebesgue measure on $\mathcal{B}(\mathbb{R})$. Show that $\mu \ll \nu$ and find the Radon-Nikodym derivative $\frac{d\mu}{d\nu}$.
(Hint. Before understanding the absolute continuity relationship between $\mu$ and $\nu$, try to understand whether or not we have $\mathrm{m} \ll \nu$.)

(3) (15+15 points) In this question, you shall consider the product measure space

$$(\mathbb{R}\times\mathbb{R},\ \mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R}),\ \mathrm{m}\times\mu)$$

where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma$-algebra of $\mathbb{R}$, $\mathrm{m}$ is the Lebesgue measure on $\mathbb{R}$ and $\mu : \mathcal{B}(\mathbb{R}) \to [0,\infty]$ is the measure given by

$$\mu(A) = \sum_{\substack{n\in A \\ n\in\mathbb{Z}}} \frac{1}{2^{|n|}}$$

a) Show that the set

$$D = \{(x,y) \in \mathbb{R}\times\mathbb{R} : \ x^2+y^2 \leq 2, \ \text{and,} \ x\in\mathbb{Q} \text{ or } y \in \mathbb{Q}\}$$

is an element of the product $\sigma$-algebra $\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})$.

b) Find the area of $D$ relative to the measure $\mathrm{m}\times\mu$, that is, find $\iint_D 1 \, d(\mathrm{m}\times\mu)$.

(4) (10+10 points) Prove the following statements.

a) Let $\mathrm{m}^*$ be the Lebesgue outer measure on $\mathbb{R}$. Let $A, B \subseteq \mathbb{R}$. If $A$ is non-Lebesgue measurable and $\mathrm{m}^*(A\Delta B) = 0$, then $B$ is non-Lebesgue measurable.

b) There exists a dense subset of $\mathbb{R}$ of Lebesgue measure 2023.