Topology TMS Exam
(JUSTIFY YOUR ANSWERS)

Denote the closure and the interior of a set C in a topological space by $\overline{C}$ and C° respectively.

1- (7+6+6+6 points) Prove/disprove the followings:

(a) If X and Y are topological spaces, $A \subseteq X$ and $B \subseteq Y$, then $(A \times B)^\circ = A^\circ \times B^\circ$ in $X \times Y$.

(b) Every subspace of a Hausdorff space is Hausdorff.

(c) An infinite set X with the finite complement topology is metrizable.

(d) If A is connected then its interior A° is connected.

2- (9+8+8 pts) (a) Show that $\phi = \{[a, b] \mid a \in \mathbb{Q} \text{ and } b \in \mathbb{R} \setminus \mathbb{Q}\}$ is a basis for a topology τ on $\mathbb{R}$.

(b) Show that the interval $(\pi, 5)$ is open in τ.

(c) Show that $\mathbb{R}$ with the topology τ is not connected.

3- (5+20 pts)

(a) Define the compactness of a topological space.

(b) Show that if X is topological space and if there is an infinite squence $A_1, A_2, A_3, \ldots$ of closed subsets of X such that

- $A_{n+1} \subsetneq A_n$ for every $n \geq 1$, and
- $\bigcap_{n=1}^{\infty} A_n = \emptyset$,

then X is not compact.

4- (25 pts) Let (X, d) be a metric space. For $x \in X$ and $A \subseteq X$, the distance between x and A is defined to be

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Prove that $d(x, A) = 0$ if and only if $x \in \overline{A}$.