20 February 2004

Graduate Preliminary Examination
Topology
Duration: 3 hours

1. Consider a topological space X, Y and a continuous map $f : X \to Y$.

a) Prove that $\overline{f^{-1}(B)} \subseteq f^{-1}(\overline{B})$ for any subset B of Y.

b) Suppose that f is also closed and surjective. Prove that $\overline{B} = f(\overline{f^{-1}(B)})$ for any subset B of Y.

c) Suppose that X is metrisable and f is a closed, surjective and continuous map. Prove that for any subset B of Y and any $y \in \overline{B}$ there exists a sequence $y_n \in B$ such that $\lim\, y_n = y$.

2.

a) Is the intersection of two dense subsets in a topological space always dense?

b) Let X be a topological space. Prove that the intersection of two open dense subsets of X is open and dense.

c) If $\tilde{\mathcal{H}}$ is the family of open dense subsets in X, prove that $\mathcal{H} = \tilde{\mathcal{H}} \cup \{\phi\}$ is a topology on X.

d) Let $\hat{X}$ be the topological space which consists of the set X with the topology $\mathcal{H}$ on it. Prove that a function $f : \hat{X} \to \mathbb{R}$ is continuous iff it is constant.

3. Let f be a continuous mapping of the compact space X onto the Hausdorff space Y. Show that any mapping g of Y into Z for which $g \circ f$ is continuous must itself be continuous.

4. Consider the cyclinder $S^1 \times I$ where S^1 the unit circle in $\mathbb{R}^2$ and $I = [0, 1]$. Identify $S^1 \times \{1\}$ to a point i.e. define an equivalence relation

$\sim$ on S^1 by letting $(u, 1) \sim (v, 1)$ for all $u, v \in S^1$ and letting all other elements in $S^1 \times [0, 1]$ be related only to itself. Show that the quotient space. $(S^1 \times I)/\sim$, the so called cone on S^1, is homomorphic to the unit disc D^2 in $\mathbb{R}^2$.