METU - Mathematics Department
Graduate Preliminary Exam-Spring 2008

Topology

1. Consider the real line $\mathbb{R}$ with the usual topology. Let $\sim$ be the equivalence relation defined by
$$x \sim y \text{ if and only if } x - y \in \mathbb{Q}.$$
Show that the quotient space $\mathbb{R}/\sim$ has uncountable number of elements and that its topology is trivial.

2. A **discrete valued map** on a topological space X is a continuous map $X \to D$ into a discrete topological space D. **Show that**

   a) X is connected if and only if every discrete valued map on X is constant.

   b) the statement "d(p) =d(q) for every discrete valued map d on X" defines an equivalence relation on X and that the corresponding equivalence classes are closed subsets of X.

3. Let X and Y be two topological spaces and let $X \times Y$ be given the product topology.

   a) Suppose K is a compact subset of X and $A \subset X \times Y$ is an open set such that for some $y \in Y$, $K \times \{y\} \subset A$. Show that y has a neighborhood $U \subset Y$ such that $K \times U \subset A$.

   b) (i) Suppose X is compact. Prove that the projection $\pi : X \times Y \to Y$ is a closed map.

   (ii) Give an example to show that in (i) the *compactness* assumption is essential.

4. Let X be a Hausdorff topological space and let $X^* = X \cup \{\infty\}$, where ∞ is an ideal point not in X. Consider the following collection Ω^* of subsets of X^*

   (i) open sets in X

   (ii) sets of the form $X^* - S$ where S is a compact subset of X.

Prove the following statements for the topological space (X^*, Ω^*) (**do not prove** that Ω^* defines a topology on X^*).

a) (X^*, Ω^*) is compact.

b) If X is locally compact, then X^* is Hausdorff.

c) A continuous map $f : X \to Y$ between Hausdorff topological spaces extends to a map $f^* : X^* \to Y^*$, which is continuous if f is proper (that is, the inverse image under f of every compact subset of Y is compact).

d) If X, and Y are locally compact Hausdorff spaces and if $f : X \to Y$ is proper and continuous, then f is a closed map.