METU MATHEMATICS DEPARTMENT
TMS EXAM IN TOPOLOGY

FEBRUARY 18, 2016

1. Parts are unrelated!

9 a) Construct a topology τ on the interval $[0,1)$ so that it becomes homeomorphic to the unit circle

$$S^1 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$$

with the topology inherited from the standard topology of $\mathbb{R}^2$. Find a homeomorphism $f : [0,1) \to S^1$, where $[0,1)$ has the topology τ.

8 b) Show that any bijection $\phi : [0,1] \to [0,1)$ is discontinuous at infinitely many points, where both interval are equipped with the standard topology inherited from the real line.

8 c) Find a homeomorphism $\psi : [0,\infty) \to (0,1]$.

2. A continuous map between two topological spaces is called proper if the preimage of any compact set is compact. Parts are unrelated!

8 a) Is there a proper map $f : \mathbb{R} \to [0,1]$, where $[0,1]$ has its standard topology? Prove your answer.

8 b) Show that $g : \mathbb{R} \to \mathbb{R}$, $g(x) = x^2$, is proper.

9 c) Let $h : X \to Y$ be a continuous map of Hausdorff topological spaces, where X is compact. Show that h is proper.

3.

10 a) Define an equivalence relation on $\mathbb{R}^2$ (equipped with the standard topology) as follows:

$$(x_0, y_0) \sim_1 (x_1, y_1) \ \Leftrightarrow x_0^2 + y_0 = x_1^2 + y_1 .$$

Show that the quotient space $\mathbb{R}^2/\sim_1$ is homeomorphic to the real line with its standard topology.

15 b) Instead of the above equivalence relation $\sim_1$ in Part (a) consider the relation $\sim_2$ defined as

$$(x_0, y_0) \sim_2 (x_1, y_1) \ \Leftrightarrow x_0^2 + y_0^2 = x_1^2 + y_1^2 .$$

Is the quotient space $\mathbb{R}^2/\sim_2$ homeomorphic to $\mathbb{R}^2/\sim_1$? Prove your answer!

4. Let (X,d) be a metric space. A function $f : X \to X$ is called an isometry if $d(x,y) = d(f(x), f(y))$, for all $x, y \in X$.

8 a) Prove that if (X,d) is compact and connected then any isometry $f : X \to X$ is a homeomorphism.

9 b) Prove that any isometry $f : \mathbb{R} \to \mathbb{R}$ is a homeomorphism, where the real line has its standard absolute value metric.

8 c) Find an isometry $f : [0,\infty) \to [0,\infty)$, which is not a homeomorphism.