Topology TMS Exam
(JUSTIFY YOUR ANSWERS)

1- (9+8+8 points)
(a) Show that

$$\mathcal{B} = \{[a, b) \mid a \text{ and } b \text{ are real numbers with } a < b\}$$

is a basis for a topology on $\mathbb{R}$, called the lower limit topology and denoted by $\mathbb{R}_\ell$.
(b) Show that the lower limit topology is finer that the usual topology on $\mathbb{R}$.
(c) Determine whether $[1, 2]$ is connected in $\mathbb{R}_\ell$.

2- (9+8+8 pts)
Let X and Y be two topological spaces. Prove or disprove the following statements:
(a) If X and Y Hausdorff, then $X \times Y$ is Hausdorff.
(b) If $p : X \to Y$ is a quotient map and if X is Hausdorff, then Y is Hausdorff.
(c) If $p : X \to Y$ is a quotient map and if Y is Hausdorff, then X is Hausdorff.

3- (5+15+5 pts)
(a) Define the compactness of a topological space.
(b) Show that if X is topological space and if there is an infinite squence $A_1, A_2, A_3, \ldots$ of nonempty closed subsets of X such that

- $A_{n+1} \subset A_n$ for every $n \geq 1$, and
- $\displaystyle\bigcap_{n=1}^{\infty} A_n = \emptyset$,

then X is not compact.
(c) Use (b) to show that $[0, 5)$ is not compact with the standard topology induced from $\mathbb{R}$.

4- (6+14+5 pts)
(a) Give the definition of a first countable and a second countable topological space.
(b) Prove that every compact metric space is second countable.
(c) Give an example of a topological space which is first countable but not second countable.