1- (9+8+8 points)
(a) Show that 
\[ B = \{ [a, b] \mid a \text{ and } b \text{ are real numbers with } a < b \} \]
is a basis for a topology on \( \mathbb{R} \), called the lower limit topology and denoted by \( \mathbb{R}_L \).
(b) Show that the lower limit topology is finer that the usual topology on \( \mathbb{R} \).
(c) Determine whether \([1, 2]\) is connected in \( \mathbb{R}_L \).

2- (9+8+8 pts)
Let \( X \) and \( Y \) be two topological spaces. Prove or disprove the following statements:
(a) If \( X \) and \( Y \) Hausdorff, then \( X \times Y \) is Hausdorff.
(b) If \( p : X \rightarrow Y \) is a quotient map and if \( X \) is Hausdorff, then \( Y \) is Hausdorff.
(c) If \( p : X \rightarrow Y \) is a quotient map and if \( Y \) is Hausdorff, then \( X \) is Hausdorff.

3- (5+15+5 pts)
(a) Define the compactness of a topological space.
(b) Show that if \( X \) is topological space and if there is a infinite sequence \( A_1, A_2, A_3, \ldots \) of nonempty closed subsets of \( X \) such that
\[ \cdot A_{n+1} \subset A_n \text{ for every } n \geq 1, \text{ and} \]
\[ \cdot \bigcap_{n=1}^{\infty} A_n = \emptyset, \]
then \( X \) is not compact.
(c) Use (b) to show that \([0, 5)\) is not compact with the standard topology induced from \( \mathbb{R} \).

4- (6+14+5 pts)
(a) Give the definition of a first countable and a second countable topological space.
(b) Prove that every compact metric space is second countable.
(c) Give an example of a topological space which is first countable but not second countable.