Preliminary Exam - February 2025

Topology

1) Let $f, g : X \to Y$ be two continuous functions and $h : X \to Y$ be defined by

$$h(x) = \begin{cases} f(x) & x \in A \\ g(x) & x \in X \setminus A \end{cases}$$

where Y is T_2 (Hausdorff) and $A \subseteq Y$ with $p \in bd(A)$ (boundary of A).

a) Show that the subset $\{x | f(x) = g(x)\}$ of Y is closed.

b) Show that h is continuous at p if and only if $f(p) = g(p)$.

c) Describe the subset $\{x | h \quad \text{is continuous at} \quad x\}$ of X.

2) Let X be a connected, compact Hausdorff (T_2) space

a) Show that the subset $Q(p) = \{O | p \in O, O \quad \text{is open and closed subset of} \quad X\}$ is connected.

b) Let $p \in O$ and $C(p, O) = U \{C | p \in C, \quad C \subseteq O, C \quad \text{is connected}\}$

where O open subset of X and $O \neq X$, Show that $\overline{C(p, O)} \bigcap bdO \neq \oslash$ (closure of $C(p, O)$).

3) Let X be a compact space and $\leq$ be a partial order on X such that $\{(x, y) | x \leq y\}$ is closed subset of $X \times X$

a) Show that the subset $F(q) = \{p | q \leq p\}$ is closed for $q \in X$.

b) Let A be closed subset of X such that for any $x, y \in A$ there is $z \in A$ such that $x \leq z, y \leq z$. Show that A has $\leq -$ greatest element.

4) Let O be an open subset of a product space $X \times Y$. Show that $P_X(O)$ is an open subset of X where $P_X(p, q) = p$ for each $(p, q) \in X \times Y$.

5) Let $f : X \to Y, \quad g : Y \to Z$ be functions such that f is continuous surjection and the composition $gof : X \to Y$ is closed (images of closed sets are closed sets under gof). Show that g is closed.

6) Let X and Y be a connected (path connected) infinite topological space. Show that $X \times Y \setminus \{(p, q)\}$ is connected (path connected) for $p \in X, q \in Y$.