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METU Department of Mathematics

Graduate Preliminary Exam	
Acad. Year : <i>2026</i> Semester : <i>Spring</i> : Date : <i>February 12, 2026</i> Time : <i>10:00</i> Duration : <i>180 minutes</i>	Full Name (USE CAPITAL LETTERS): <input type="text"/> Student ID: <input type="text"/>
5 QUESTIONS TOTAL 100 POINTS	Signature: <input type="text"/>

Q1 (20 pts) If p and q are primes, then show that any group of order p^2q is solvable.

Q2 (20 pts) Let G be an infinite group containing an element $x \neq e$, having only finitely many conjugates. Prove that G is not simple.

Q3 (20 pts) Let G be a finite group with an automorphism ϕ such that $\phi(x) = x$ if and only if $x = e$.

(a) Show that every element of G can be written as $x^{-1}\phi(x)$.

(Hint: Consider the map $\phi : G \rightarrow G$ with $\phi(x) = x^{-1}\phi(x)$).

(b) Suppose $\phi^2(x) = x$ for all $x \in G$. Prove that $\phi(x) = x^{-1}$ for all $x \in G$.

Q4 (20 pts) Suppose that a is a non-zero non-unit element of an integral domain R .

(a) Prove that the ideal (a, x) in $R[x]$ is not principal.

(b) Use (a) to show that if F is a field, then $F[x, y]$ is not a PID.

Q5 (20 pts) Let R be a principal ideal domain. Show that every proper ideal of R is a product $P_1P_2 \cdots P_n$ of maximal ideals, which are uniquely determined up to order.