METU Department of Mathematics

Q1 (20 *pts*) If p and q are primes, then show that any group of order p^2q is solvable.

Q2 (20 *pts*) Let G be an infinite group containing an element $x \neq e$, having only finitely many conjugates. Prove that G is not simple.

Q3 (20 *pts*) Let G be a finite group with an automorphism ϕ such that $\phi(x) = x$ if and only if $x = e$.

(a) Show that every element of G can be written as $x^{-1}\phi(x)$.

(Hint: Consider the map $\phi : G \to G$ with $\phi(x) = x^{-1}\phi(x)$.)

(b) Suppose $\phi^2(x) = x$ for all $x \in G$. Prove that $\phi(x) = x^{-1}$ for all $x \in G$.

Q4 (20 *pts*) Suppose that a is a non-zero non-unit element of an integral domain R.

(a) Prove that the ideal (a, x) in $R[x]$ is not principal.

(b) Use (a) to show that if F is a field, then $F[x, y]$ is not a PID.

Q5 (20 *pts*) Let R be a principal ideal domain. Show that every proper ideal of R is a product $P_1 P_2 \cdots P_n$ of maximal ideals, which are uniquely determined up to order.