February, 2026

Graduate Preliminary Examination
Numerical Analysis I
Duration: 3 Hours
(4 Questions in 2 Pages)

1. Let $A \in \mathbb{R}^{n\times n}$ be a nonsingular matrix and let $b \neq 0$ be a given vector. Suppose that x and $\hat{x}$ satisfy

$$Ax = b, \qquad (A + \delta A)\hat{x} = b,$$

where δA denotes a perturbation of the matrix A.

(a) Show that if $\|A^{-1}\delta A\| < 1$, then $\dfrac{\|x-\hat{x}\|}{\|x\|} \leq \dfrac{\|A^{-1}\delta A\|}{1 - \|A^{-1}\delta A\|}$.

(b) Show that if $\|A^{-1}\|\,\|\delta A\| < 1$, then

$$\frac{\|x-\hat{x}\|}{\|x\|} \leq \frac{\kappa(A)\,\dfrac{\|\delta A\|}{\|A\|}}{1 - \kappa(A)\,\dfrac{\|\delta A\|}{\|A\|}},$$

where $\kappa(A)$ is the condition number of A with respect to the chosen matrix norm.

(c) Consider

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 10^{-2} \end{pmatrix}, \qquad \delta A = \begin{pmatrix} 0 & 0 \\ 0 & 10^{-4} \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Using the bound obtained in part (b) and the matrix infinity norm, show that

$$\frac{\|x-\hat{x}\|_\infty}{\|x\|_\infty} \leq 1.1 \times 10^{-2}.$$

2. Let $A \in \mathbb{C}^{n\times n}$ be nonsingular. Consider the sequence of matrices $\{X_k\}$ defined by

$$X_{k+1} = X_k + X_k(I - AX_k), \quad k = 0, 1, 2, \ldots$$

and assume that the initial matrix X_0 satisfies $\rho(I - AX_0) < 1$, where $\rho(\cdot)$ denotes the spectral radius.

(a) Show that $X_k \to A^{-1}$ as $k \to \infty$.

(b) Let $b \in \mathbb{C}^n$ and define the iterative method for solving $Ax = b$ using the matrices X_k:

$$x^{(k+1)} = x^{(k)} + X_k(b - Ax^{(k)}).$$

(i) Show that the error $e^{(k)} = x^{(k)} - A^{-1}b$ satisfies

$$e^{(k+1)} = (I - AX_k)e^{(k)}.$$

(ii) Using part (a), prove that $x^{(k)} \to A^{-1}b$ for any initial vector $x^{(0)}$.

(c) Consider the matrices AX_k.

(i) Show that $AX_k \to I$ as $k \to \infty$.

(ii) Explain how the singular values of X_k approximate those of A^{-1} as k increases.

3. Let A be a real symmetric matrix with a diagonal Jordan canonical form

$$P^{-1}AP = D = diag(\lambda_1, \lambda_2, \cdots, \lambda_n)$$

for some non-singular matrix P containing eigenvectors of A as columns corresponding to the eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$. The pair (λ, x) is an approximate eigenvalue and corresponding eigenvector of A with $\|x\|_2 = 1$. Show that the residual vector

$$r = Ax - \lambda x$$

satisfy

$$\min_{1 \le i \le n} |\lambda - \lambda_i| \le \|r\|_2.$$

Hint: Use l_2- matrix norm of a diagonal matrix as the maximum entry in magnitude.

4. A matrix $A \in \mathbb{Z}^{4\times 4}$ is being QR–factorized. After one Householder transformation using the matrix Q_1 generated by v, one has found

$$A_2 = Q_1 A = 7 \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & -\frac{3}{5} & \frac{3}{5} & \frac{1}{5} \\ 0 & -\frac{4}{5} & \frac{4}{5} & -\frac{7}{5} \end{pmatrix}, \qquad v = \frac{w}{\|w\|_2}, \qquad w = \begin{pmatrix} -10 \\ 0 \\ -2 \\ -6 \end{pmatrix}.$$

(a) Determine the original matrix A. You may use that

$$\frac{2\, w^T A_2}{w^T w} = \left[-1, \frac{8}{5}, -\frac{8}{5}, \frac{9}{5}\right].$$

(b) Determine the upper triangular matrix R such that $A = QR$ using Householder transformations. Give also the vectors v_2 and v_3 which generate the Householder matrices Q_2 and Q_3. You are *not* required to compute Q.