# Real Analysis Preliminary Exam
# Feb 2026

$m$ **refers the Lebesgue measure.**

1. (12+13 pts) Let $f_n \in L^1(\mathbb{R}, m)$ such that $f_n \to f$ a.e. and $\sup_n \int_{\mathbb{R}} |f_n| dm < \infty$. For each of the following statements, if the statement is true, prove it, while if false give a counterexample.

   (a) $f \in L^1$

   (b) $\int_{\mathbb{R}} f_n dm \to \int_{\mathbb{R}} f dm$.

2. (25 pts) Let $0 < \delta < 1$ and suppose $A$ is Borel measurable subset of $\mathbb{R}$. Prove that if

$$m(A \cap I) \leq (1 - \delta) m(I)$$

   for all interval $I$ then $m(A) = 0$.

3. (25pts) Let $f \in L^2$ and $||f||_2 \leq 1$. Show that

$$\int_0^1 \frac{dx}{1 + |f(x)|^2} \geq \frac{1}{2}.$$

4. (25pts) Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function. Prove that

$$\int_{-\infty}^{\infty} |f(x)|\, dx = \int_0^{\infty} m\big(\{x \in \mathbb{R} : |f(x)| \geq t\}\big)\, dt,$$

   where $m$ denotes Lebesgue measure.

   **Hint:** Use Tonelli.